# Priced Diversifiable Risk and Other Unknown Corollaries of the Capital Asset Pricing Model

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**Abstract**

The capital asset pricing model (CAPM) is a mainstay of finance logic but important principles construed from CAPM are misleading. When interpreted correctly, CAPM implies that there is practically no aspect of a risky venture that does not influence its cost of capital (up or down). Risky cash payoffs that seem "idiosyncratic" can have high *ex ante* beta, despite having extremely low correlation with the market. This occurs when they have sufficiently low payoff mean relative to covariance. Also, payoff risks that are completely "diversified away", and unpriced at market-level, can be priced heavily at the individual firm or asset level.

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# 1 Introduction

Portfolio theory and its logical offshoot - the capital asset pricing model (CAPM) - are Nobel prize winning inventions in financial economics, and for 40 years have been the basis of every finance course in universities. The CAPM and its implications are pedagogical building blocks across finance, and have enchanted practitioners and theorists alike. Although not widely supported as an empirical relationship, the CAPM is universally upheld for its conceptual elegance and economic intuitions.[1] Applications of CAPM are standard in all facets of investment under uncertainty, including corporate valuation, capital budgeting, funds management, and fund manager performance evaluation. Peripheral but socially and economically important applications include tariff or rate setting in the regulated utilities.[2]

In economics, the CAPM is understood as an equilibrium model which can be treated upon suitable assumptions as either:

(i) a positive (descriptive) model of how markets with homogenous beliefs arrive at the prices of a set of risky assets, all weighed by their marginal effects on the risk and return of an optimal portfolio, or

(ii) a normative model of how an individual with personal (subjective) beliefs should value one risky asset relative to the others in a personally optimal investment portfolio.

In either case, the logic and insights from the CAPM are essentially the same. The problem, however, is that finance in its zeal has instructed generations of students in a false or overstated understanding of what the CAPM "tells us". From the time of its invention[3] and the finance revolution of the 1960-70s,[4] the CAPM brought a Kuhnian shift in both real world investment practice and the textbook theory of *ex ante* equilibrium asset values. Prompted perhaps by the resistance it faced

[1] The CAPM "lives on" (Levy 2010, 2011). See Welch (2014, Ch.9) on the dominance of CAPM and its intuition over other valuation models in corporate, regulatory and legal (torts) contexts.

[2] Chartoff et al. (1982) describe how US regulators took up the CAPM.

[3] Interesting history on CAPM development is provided by Sullivan (2006) and French (2003).

[4] The finance revolution was based on three of four big ideas, all essentially to do with imposing a rational law of "no arbitrage" or no unfairly priced risk. See Bernstein (1992) for the fascinating story of how the field of financial economics arose and quickly came to dominate large parts of business practice and thinking.

from old ways of thinking and personal rivalries in practice and academe, CAPM interpretations and logical insights were exaggerated in ways that made them more definite and more extraordinary than is justifiable by valid CAPM logic.

The least important errors in common CAPM discourse might be explained away as just convenient abuse of language. One such example is the way that virtually all textbooks treat the terms "market risk" and "undiversifiable risk" as synonyms, when in fact any CAPM market retains some diversifiable risk in its residual or market risk, for the reason that some amount of residual diversifiable risk "pays its way" in marginal expected return or utility. That rational compromise between risk and return is one of the firmest ideas in economics, and is effected implicitly by the CAPM equilibrium.

Other errors in how we understand the CAPM are errors of "CAPM logic" rather than language, and have brought about decades of logically false commercial practice (and expert witness testimony). One such fundamental error comes from the way that we define the "risk class" of an asset by its CAPM returns beta (or just its "beta", as every financial economics student knows it). It is possible with just basic algebra to show that under CAPM the beta of a firm or asset is a function of not just its risk or payoff uncertainty (statistical variability) but also of its payoff mean. A shift in mean with no change in risk changes beta. This simple point goes unmentioned and repeatedly contradicted in finance textbooks.

A correct understanding is that if a firm benefits from say a regulatory decision bringing it a higher *ex ante* (forward-looking) expected payoff, with no change in its payoff variance or covariance with the market, it now has a new (lower) beta and a new (lower) CAPM cost of capital or expected return. The effect of the mean payoff on beta is easy to prove mathematically. This effect is listed below as just one fundamental insight in a battery of valuable yet apparently "unknown" CAPM corollaries. The benefit of dispelling entrenched CAPM myths is that for every long standing logical error that is demonstrated, a better understanding of CAPM and a new and logically correct insight comes in its place.

## 1.1 Dispelling CAPM Myths

In a chapter of their celebrated textbook, *Theory of Finance*, Fama and Miller (1972) developed a theory of capital budgeting under uncertainty by constructing the capital asset pricing model (CAPM) as a forward-looking decision tool.[5] By extending Fama's interpretation of CAPM as a strategic decision tool, based on the CAPM in its "pricing" (or certainty equivalent) form rather than its better known "returns" form, it is easily proved that important economic principles held to have their basis in CAPM are false on their very own terms, as a matter of no more than CAPM logic. Moreover, fundamental aspects of what is said about the CAPM are not only misleading, they effectively hide contrary economic insights which, when understood, are at least as appealing and useful as those dispelled.

To spark my claim that the CAPM is misrepresented, I cite another respected finance textbook, chosen for its depth, clarity and wit, and widely regarded as a benchmark. This book, *Corporate Finance*, by Brealey, Myers and Allen (2014) tells us not to be "fooled by diversifiable risk". It says that a firm can take on new products that might fail outright, drill oil wells at random, sell new and possibly untested drugs, and perhaps do all of these things at once, while still not adding to its CAPM cost of capital. Why is that? The answer drummed into finance students is that these activities are independent of one another and moreover of "the market". In finance language they are "unique", "firm-specific" or "idiosyncratic" risks, which by the law of large numbers cancel one other out over a sufficiently large diversified portfolio.

If this is right, where does it end? Does it mean that the firm can try to build a time machine or clone the Tasmanian tiger, since that would be idiosyncratic? And how can the market embrace firms that embark on long shots like an untested patent or hydrogen car without any firm paying any penalty in its "risk-adjusted" discount rate? That would allow firms as much unpriced risk as they like, all absorbed by the market without other firms in the market wearing any cost either.

A valid interpretation of CAPM, illustrated in what follows by equations and numerical examples, contradicts the usual CAPM tenet that investment projects which are largely unaffected by "the market", or general market influences, can not (in fact, must not) attract a high CAPM discount rate.

[5] The theory in Fama and Miller (1972) was clarified by Fama (1977) and traces to the *ex ante* decision-theoretic interpretation of CAPM in Sharpe (1964). It is the basis of Bayesian portfolio theory and Bayesian treatment of parameter uncertainty. References include Markowitz (1991, p.470). See also Shanken (1987), Harvey and Zhou (1990), Coles and Loewenstein (1988), and Rachev, Hsu, Bagasheva and Fabozzi (2008).

## 1.2 List of Correct "Unknown" CAPM Corollaries

This list rests on standard CAPM assumptions. Assets are valued or "priced" under an assumed utility function and an exogenous joint distribution of their one-period money payoffs. Each individual asset (project or whole firm) is small compared to the reference set of all assets, so its CAPM parameters (means and covariances) have negligible effect on the aggregate market CAPM parameters.

(i) Even the most "idiosyncratic" looking activities with near zero correlation with the market can have high "beta" and accordingly high CAPM cost of capital. Paradoxically, an asset with positive CAPM price and zero correlation with every other asset can have a discount rate under CAPM approaching infinity.

(ii) The beta of a risky payoff is a function of both its payoff covariance (with the market aggregate) and payoff mean, so beta is not a measure purely of statistical variability or "risk". A higher mean payoff is enough of itself to reduce beta.

(iii) Beta is a sufficient statistic with respect to the CAPM discount rate, but the minimal sufficient statistic is the ratio of payoff covariance to payoff mean. This ratio proves so useful that it is given a name, the "Fama ratio", after Fama (1977) who first suggested its role in CAPM relationships.

(iv) Risky activities that are perfectly diversified by the market, in the sense that the sum of their payoffs is a known constant and adds zero to the market payoff variance, can nonetheless be discounted heavily at the individual asset or firm level, contrary to the usual finance rule that "there is no reward for holding diversifiable risk". When the sum of two risky payoffs is risk-free, their price-weighted discount rates balance out such that their sum is discounted at the risk-free rate.

(v) If an asset's Fama ratio is unchanged, any change in the market risk premium brings a change in the asset's beta and when combined these joint changes in asset beta and market risk premium cancel one another out, leaving the asset's expected return unchanged.

(vi) Even a highly risk averse market leaves some diversifiable risk undiversified.

Further diversification reduces *ex ante* expected utility. Since market risk is partly diversifiable, the terms "undiversifiable risk" and "market risk" are strictly not synonyms. The common statement that "only market risk is priced" is tautologous. "Market risk" is merely the name given to risk that the market chooses to carry

(vii) A more risk averse market is characterized in equilibrium by a *higher* expected market return and a *higher* market return variance. Surprisingly, both parameters are higher in a more risk averse market, however the Sharpe ratio, which captures the equilibrium ratio of market risk-premium to market risk, is also higher.

(viii) The equilibrium Sharpe ratio of the market is the product of the market's payoff variance and the utility function parameter that represents the market's aversion to payoff variance. Remarkably, the exogenous risk-free rate has no effect on the CAPM equilibrium Sharpe ratio.

(ix) Given an exogenous joint payoffs distribution, a more risk averse market has both a higher Sharpe ratio and a lower aggregate market price ("market cap"). The Sharpe ratio is not a complete picture of the market's risk aversion. If there are two markets with different payoff distributions, the market with the higher Sharpe ratio can be the less risk averse.

(x) If a sufficiently large number of firms all take on "zero beta" investments the market risk premium might change, and can increase or decrease, all depending on how the market's expected payoff (or more specifically, its Fama ratio) is affected

(xi) Two investment projects or risky payoffs ("lotteries") that are perfectly positively correlated have equal betas and hence discount rates, *if and only if* they are directly proportional. Proportional cash flows are sufficient for two projects to have the same beta and discount rate, but not necessary. The necessary condition is that the payoffs have the same Fama ratio.

(xii) Any asset with positive CAPM price approaching zero, and positive covariance with the market, must necessarily have a CAPM discount rate or cost of capital approaching infinity.

(xiii) "Cash flow news" affects the estimated expected payoff, and by that fact alone is also "discount rate news"; i.e. the "numerator" (expected payoff) affects its own "denominator" (discount rate).

(xiv) Information risk, or the risk that a signal is false or unreliable, is priced risk. Bayesian allowance for information risk can *reduce* the firm's cost of capital.

(xv) A randomized strategy of choosing between two discrete payoff distributions by spinning a wheel creates a new payoff distribution for which the cost of capital can be either higher or lower than the probability-weighted average cost of capital of the assets being mixed. Pure chance events affecting a payoff can thus be "priced".

# 2 Fama's Ratio is Minimal Sufficient

The expected return on a business venture or risky asset depends on its payoff mean as well its payoff variance, and is therefore not purely about project risk. To see this result, write the CAPM in pricing rather than returns form, as in Luenberger (1998).

$$P_j = \frac{E[V_j] - c\,\mathrm{cov}(V_j, V_M)}{R_f}, \qquad (R_f \equiv 1 + r_f) \tag{1}$$

where $P_j$ is the price of asset $j$, $E[V_j]$ is its expected payoff, $\mathrm{cov}(V_j, V_M)$ is its payoff covariance with the market payoff $V_M$ ($\equiv \sum V = V_1 + V_2 + ....$), $r_f$ is the risk-free interest rate and $c$ is an exogenous constant capturing the market's aversion to payoff variance. If (1) is derived from an assumed utility function (e.g. Huang and Litzenberger, 1998; Lambert et al. 2007; Barucci 2003), then $c$ has a given theoretical value set by the risk aversion parameter in that function.[6]

Under the assumption of forward-looking estimates of the exogenous primitives $E[V_M]$ and $\mathrm{var}(V_M)$, the effect of the market's innate risk aversion $c$ surfaces in the

[6] For example, if payoffs are joint normal and the market's utility function is $u(x) = -\mathrm{Exp}(-\tau x)$, then $c = \tau$, which happens to be the constant coefficient of absolute risk aversion. In the other common case justifying mean-variance methods, the market utility function is quadratic, $u(x) = x - \frac{b}{2}x^2$ $(b > 0)$, and it can be shown that $c = b/\left\{1 - b\left(E[V_M] + (W - P_M)\,R_f\right\}\right.$, where $W$ is the total market wealth in all assets including the risk-free.

endogenous market price of the aggregate of all assets, $P_M(\equiv \sum P = P_1 + P_2 + ...)$, since it can be seen mathematically from (1) that $c = (E[V_M] - P_M R_f)/\text{var}(V_M)$. Specifically, exogenous $c$ drives endogenous $P_M$, and is therefore the driver of the market parameter $(E[V_M] - P_M R_f)/\text{var}(V_M)$. (An empirical estimate of $c$ is given later in this paper.)

Following from the pricing or "certainty equivalent" form of CAPM, written as (1), the forward-looking expected return on asset $j$ is

$$E[R_j] = \frac{E[V_j]}{P_j} = \frac{E[V_j]\ R_f}{E[V_j] - c\ \text{cov}(V_j, V_M)} = R_f \left[1 - c\left(\frac{\text{cov}(V_j, V_M)}{E[V_j]}\right)\right]^{-1}. \quad (2)$$

For typical assets with positive payoff mean and covariance parameters, the firm's expected return or "cost of capital" $E[R_j]$ is: (i) increasing in its payoff covariance, as is well known, and (ii) decreasing in its mean payoff, as is widely unknown.

The effect of the mean payoff on $E[R_j]$ reveals that so-called "cash flow news" is *ipso facto* also "discount rate news". More specifically, when a risky future cash flow is valued by "risk adjusted discounting", the discount rate implied by the CAPM for its "risk class" increases when the expected cash flow increases.

From (2) it follows that assets (including whole firms) belong into the same CAPM "risk class", and have the same cost of capital, *if and only if* they have the same ratio

$$\digamma_j = \frac{\text{cov}(V_j, V_M)}{E[V_j]}.$$

I call this Fama's ratio after Fama (1977) where it was identified explicitly as the source of difference between different firms' expected returns (see the later quote).

The usual and much better known rule is that assets are in the same risk class when they have the same returns beta, $\beta_j = \text{cov}(r_j, r_M)/\text{var}(r_M)$. Fama (1977) noted that the Fama ratio is connected to beta, however the formal relationship between these parameters does not seem to have been specified previously in the literature, and Fama's demonstration seems to have been oddly overlooked.

Both rules are correct. Their only difference is that one is stated in terms of asset payoffs and the other in terms of returns, where returns are payoffs divided by an endogenous equilibrium price that is affected by both first and second payoff moments. Beneath the surface, those joint effects are assimilated mathematically into beta.

To show their mathematical relationship, and to reconcile the two rules, it helps to write asset beta as a function of the asset's Fama ratio, along with some essential market parameters.

The returns beta of firm $j$, defined as usual, is

$$\begin{aligned}\beta_j &= \frac{\mathrm{cov}(r_j, r_M)}{\mathrm{var}(r_M)} \\ &= \frac{\mathrm{cov}(V_j, V_M)/P_j P_M}{\mathrm{var}(V_M)/(P_M)^2} \\ &= \frac{\mathrm{cov}(V_j, V_M)}{P_j}\left[\frac{P_M}{\mathrm{var}(V_M)}\right] \qquad (3)\end{aligned}$$

Substituting for $P_j$ from (1) and rearranging gives

$$\beta_j = R_f \left(\frac{\digamma_j}{1 - c\digamma_j}\right)\left(\frac{P_M}{\mathrm{var}(V_M)}\right), \qquad (4)$$

where $P_M$ and $\mathrm{var}(V_M)$ are market level quantities that by the usual assumption are too large to be affected materially by one single asset. Note that for any asset with $P_j > 0$, it follows from (1) that $c\digamma_j < 1$. Note also that assets with zero payoff covariance with the market have zero Fama ratio and hence zero beta.

In statistical language, CAPM "beta" is a sufficient statistic but not the "minimal sufficient" statistic. It can be seen from (4) that the most reduced yet still sufficient index of an asset's risk class under the CAPM is its Fama ratio $\digamma_j$. Assets with equal $\digamma_j$ necessarily have equal beta and expected return in any mean-variance efficient market, regardless of the local exogenous market parameters $P_M$, $\mathrm{var}(V_M)$ and $c$.

The distinction between the minimal sufficient $\digamma_j$ and the merely sufficient $\beta_j$ shows in the way that beta contains information about the market in addition to, and surplus to, information about the asset of itself.

## 2.1 Beta is a Combination of Two Fama Ratios

An asset's beta is a function of its own Fama ratio and the market's Fama ratio.

Rewriting (2) as

$$E[R_j] = R_f\ [1 - c\ \digamma_j]^{-1} \qquad (R_j \equiv (1 + r_j)) \qquad (5)$$

and letting the asset in (5) be the whole market,

$$E[R_M] = R_f \; [1 - c \, \digamma_M]^{-1} \, , \tag{6}$$

where

$$\digamma_M = \frac{\mathrm{cov}(V_M, V_M)}{E[V_M]} = \frac{\mathrm{var}(V_M)}{E[V_M]}$$

is the Fama ratio of the market.

Now the returns beta of firm $j$, defined conventionally by (3), is

$$\begin{aligned} \beta_j &= \frac{\mathrm{cov}(V_j, V_M)}{\mathrm{var}(V_M)} \frac{P_M}{P_j} \\ &= \frac{\mathrm{cov}(V_j, V_M)}{E[V_j]} \frac{E[V_M]}{\mathrm{var}(V_M)} \frac{E[R_j]}{E[R_M]}. \end{aligned}$$

Substituting for $E[R_j]$ and $E[R_M]$ from (5) and (6), gives

$$\beta_j = \left( \frac{\digamma_j}{\digamma_M} \right) \left( \frac{1 - c \, \digamma_M}{1 - c \, \digamma_j} \right), \tag{7}$$

from which it follows that $\beta_M = 1$.

This insightful formula (7) is new, and shows that beta is determined fundamentally by just three parameters: the market's payoff variance aversion $c$, the market's Fama ratio and the firm's own Fama ratio, all of which are exogenous.

Some interesting previously unstated aspects of beta are implied:

(i) a firm or project has $\beta_j = 1$ if and only if it has the same Fama ratio as the market. That is, $\digamma_j = \digamma_M$, or

$$\frac{\mathrm{cov}(V_j, V_M)}{E[V_j]} = \frac{\mathrm{var}(V_M)}{E[V_M]}.$$

(ii) assuming, as is typical, that $c > 0$ and $R_f > 1$, a positive asset price $P_j > 0$ implies by (1) that $\digamma_j < 1/c$, in which case $\beta_j > 1$ if and only if $\digamma_j > \digamma_M$. Thus, to have $\beta > 1$, the asset must have a higher Fama ratio than the market, i.e. $\digamma_j > \digamma_M$.

(iii) given $E[V_j] > 0$ and $\mathrm{cov}(V_j, V_M) > 0$, $\digamma_j$ is positive, in which case $\beta_j$ is increasing in $\digamma_j$ and decreasing in $\digamma_M$ (noting that $\digamma_M$ is positive for any $E[V_M] > 0$).

(iv) although $\beta_j$ is decreasing in $\digamma_M$ for assets with $\digamma_j > 0$, the expected return $E[R_j]$ is unaffected by $\digamma_M$ (provided $\digamma_j$ is constant). Any change in $\digamma_M$ has twin effects on $E[R_j]$ and these exactly offset one another. One effect is via $\beta_j$ and the exactly compensating effect is via the market return $E[R_M]$. To understand how these effects cancel one another, consider the expected return given by the usual returns form of CAPM

$$E[R_j] = R_f + \beta_j(E[R_M] - R_f). \quad (R_f = 1 + r_f)$$

Now substitute into this CAPM equation $E[R_M]$ from (6) and $\beta_j$ from (7). The mathematical result is the reappearance of (5), showing therefore that $E[R_j]$ is not affected by changes in the *market* Fama ratio $\digamma_M$.

This independence from the market is remarkable. It says that an asset's CAPM cost of capital depends on its own payoff properties, specifically on $\digamma_j$, but not on the payoff properties of the market as a whole.

Holding $\digamma_j$ constant, any change in the market's Fama ratio $\digamma_M$ causes a change in the market cost of capital $E[R_M]$ but no change in the asset's cost of capital $E[R_j]$. Thus, an increase in market payoff risk, $\text{var}(V_M)$, caused by changes in assets other than asset $j$, can add to the market risk premium yet leave asset $j$'s risk premium unchanged.

Generally, of course, $\digamma_j$ will tend to change in conditions where $\digamma_M$ changes. For example, an increase in market payoff variance can be brought about by the covariances between many individual assets increasing. If asset $j$ is one of these, $\digamma_j$ will increase unless any accompanying increase in $E[V_j]$ happens to exactly cancel the increase in $\text{cov}(V_j, V_M)$ and leave $\digamma_j$ unchanged.

(v) in the usual circumstances where $c > 0$ , $\digamma_M > 0$, and $\digamma_j < 1/c$ (i.e. $P_j > 0$), beta $\beta_j$ is increasing in $c$ when $\digamma_j > \digamma_M$ and decreasing in $c$ when $\digamma_j < \digamma_M$. This allows the average CAPM beta to remain equal to one even when the market variance aversion $c$ changes. From (5), $E[R_j]$ is increasing in $c$ when $\digamma_j > 0$ and decreasing in $c$ when $\digamma_j < 0$.

# 3 Beta is Not Purely a Risk Measure

A common error is to assume that the returns covariance and payoff covariance are equivalent risk measures. Rather, the returns covariance is a function of both the payoff covariance and the payoff mean. Specifically, it is easily shown by substituting from (1) for $P_j$ that

$$\text{cov}(r_j, r_M) = \text{cov}(V_j, r_M)/P_j = \left[\frac{E[V_j]}{\text{cov}(V_j, r_M)} - k\right]^{-1}, \tag{8}$$

where $k = (E[r_M] - r_f)/\text{var}(r_M)$ is a market constant (driven by $c$). In the usual returns derivation of CAPM, $k$ is called "the price of risk" and summarizes the market's preferred trade-off between expected return and return variance.

The mistake is to think of the returns covariance $\text{cov}(r_j, r_M)$ in (8) as merely the payoff covariance $\text{cov}(V_j, r_M)$ divided by a constant, namely $P_j$, and hence to view the two covariances as equivalent. That way of thinking misses the fact that $P_j$ is a constant only once it has been determined endogenously by the joint effects of both the payoff covariance and the payoff mean.

There is a fundamental difference between the returns covariance - or its normalized form "beta" - and the payoff covariance. The payoff covariance is a measure purely of variability or risk, but beta is not. The error of taking these two measures as analogous is common. Fama and Miller (1972, p.298) state that payoff and returns covariances are "comparable" measures of asset or firm risk. To the contrary, a given returns covariance when normalized implies beta, but a given payoff covariance implies only the sign of beta.

# 4 A Standard Mistake

One of the entrenched mistakes in finance based on a misunderstanding of CAPM logic is the repeated claim that if a firm increases its operating leverage its net cash payoff becomes more risky and hence its beta and cost of capital increase. A clear statement follows;

> Another factor that can affect the market risk of a project is its degree of operating leverage, which is the relative proportion of fixed versus variable costs. Holding fixed the cyclicality of the projects's revenues, a

higher proportion of fixed costs will increase the sensitivity of the project's cash flows to market risk and raise the project's beta. To account for this affect, we should assign projects with an above-average proportion of fixed costs, and thus greater-than-average operating leverage, a higher cost of capital. (Berk and DeMarzo 2014, p.420)

Similarly, Brealey et al. (2014, p.226) state

> A production facility with high fixed costs relative to variable costs, is said to have high operating leverage. High operating leverage means a high asset beta.

This claim, while completely accepted in finance, is false, and indeed false under CAPM, which is remarkable given that its supposed basis is CAPM.

A quick demonstration follows. Suppose that fixed costs are increased from $F_1$ to $F_2$, with the effect that variable costs (per unit) decrease from $v_1$ to $v_2$. Let the uncertain (random) demand in units be $x$, and let its mean and covariance with the market payoff be $E[x]$ and $\text{cov}(x, V_M)$ respectively. If the unit sale price is s, then the firm's future net cash flow $V = x(s - v) - F$ had initial Fama ratio

$$\frac{\text{cov}(V, V_M)}{E[V]} = \frac{(s - v_1)\ \text{cov}(x, V_M)}{(s - v_1)E[x] - F_1}$$

and new Fama ratio

$$\frac{(s - v_2)\ \text{cov}(x, V_M)}{(s - v_2)E[x] - F_2}.$$

So the firm's Fama ratio, and hence its beta and cost of capital, will actually decrease if

$$\frac{(s - v_2)\ \text{cov}(x, V_M)}{(s - v_2)E[x] - F_2} < \frac{(s - v_1)\ \text{cov}(x, V_M)}{(s - v_1)E[x] - F_1},$$

or more simply if

$$\frac{(s - v_2)}{(s - v_1)} > \frac{F_2}{F_1}.$$

So provided that the increase in operating leverage brings a large enough reduction in variable costs per unit, relative to how much it costs in additional fixed cost, the CAPM cost of capital will decrease. Clearly, therefore, a big reduction in variable costs per unit can bring both a higher ratio of fixed to variable cost and at the same time a lower cost of capital.[7]

The mistake of equating higher operating leverage with higher beta is merely one case of how finance goes wrong by thinking of the cost of capital as simply and entirely "risk related", rather than as a fusion of payoff risk and payoff mean. To avoid this mistake, CAPM theory needs to recognize that although the CAPM discount rate can be expressed formally as a function of just the second moment of *returns*, or equivalently of just "beta", the rational CAPM discount rate is driven economically, or in terms of "fundamentals", by both the first and second moments of *payoffs*. A most convenient property of CAPM is that this synthesis of payoff moments is fully summarized by a single number - Fama's ratio.

# 5 Diversifiable Risk is Not All Diversified

The notion that the CAPM does not diversify all diversifiable risk goes gratingly against most peoples' CAPM understanding and is worth clarification. A contrary intuition, also widely accepted and long established in economic decision theory, is that diversification pays only up to a point; beyond which point, payoff risk (here payoff variance) can be reduced by further diversification but with such a drop in expected payoff that marginal expected utility is negative. Only the second of these rival intuitions is correct.

The CAPM derives from expected utility maximization. Based on an exogenous albeit subjective payoffs distribution, the CAPM produces asset prices $(P_1, P_2, ...)$ that have the unique property that the marginal expected utility obtained by selling $\$\delta$ $(\delta \rightarrow 0)$ worth of one asset to buy $\$\delta$ worth of another, or to put that $\$\delta$ in the risk-free asset, is zero. Investors could reduce the variance of the period-end wealth by holding more of those assets with lower payoff covariances, or by simply shifting more initial wealth to the risk-free asset, but this would reduce their expected utility.

So how does a market decide how far to diversify? The answer is that the market

[7] In hindsight, the "law" that higher operating leverage should always bring a higher cost of capital is not economically plausible. Its potential absurdity can be shown by taking an extreme case. Suppose that the increase in fixed costs is marginal, i.e. $(F_2 - F_1) \simeq 0$, and the savings in variable cost are huge, i.e. $v_2 \rightarrow 0$. All that has happened effectively is that the payoff distribution has a tiny bit of mass shifted very slightly to the left and much more of its mass shifted to the right. With $v_2 < v_1$, the new payoff distribution becomes first order stochastic dominant over the existing distribution when $F_2 \rightarrow F_1$, which cannot justify a higher discount rate.

does not have an explicit diversification target. Instead, it has a utility function, and implied level of risk aversion, that leads to asset prices $(P_1, P_2, ...)$ with the property described above. Since these prices are ultimately portfolio weights, they decide how diversified the market turns out to be. That level of diversification - which need not be great, since the market might not be highly risk averse - is merely a "side effect" of expected utility maximization.

A virtually unknown implication of CAPM is that a *more* risk averse market produces prices that imply (i) a higher expected market return, as might be guessed, and (ii) a *higher* market returns variance, which is more unexpected.

These results are easily shown as follows. Defining the expected market return as usual as $E[r_M] = \frac{E[V_M]}{P_M} - 1 = \frac{E[V_M]\,R_f}{E[V_M] - c\,\mathrm{var}(V_M)} - 1$, its derivative with respect to the risk aversion parameter $c$ is

$$\frac{d(E[r_m])}{dc} = \frac{E[V_M]\,\mathrm{var}(V_M)R_f}{(E[V_M] - c\,\mathrm{var}(V_M))^2},$$

which is positive, implying that $E[r_m]$ is increasing in $c$. Similarly, defining the market return standard deviation as $\sigma(r_M) \equiv \sqrt{\mathrm{var}(V_M)}/P_M$, its derivative with respect to $c$ is

$$\frac{d(\sigma(r_m))}{dc} = \frac{\mathrm{var}(V_M)^{\frac{3}{2}}R_f}{(E[V_M] - c\,\mathrm{var}(V_M))^2},$$

which is also positive, and hence the market's equilibrium standard deviation of return $\sigma(r_M)$ is increasing in its risk aversion parameter $c$.

The question begging is why or how a more risk averse market must necessarily have a higher equilibrium return risk or standard deviation. That seems to defy rationality since surely more risk averse investors would require lower returns risk?

The answer to this puzzle is twofold. First, the sum of money invested in the market is $P_M = P_1 + P_2 +$, and is decreasing in $c$, as can be seen from (1). Hence, a higher market risk aversion shows up primarily in less money being put into the market for risky assets. Second, that money amount held in the market is held at a higher ratio of return to risk. This required tradeoff between risk and required return is measured by the Sharpe ratio

$$S = \frac{E[r_M] - r_f}{\sigma(r_M)}. \tag{9}$$

With $E[r_M]$ and $\sigma(r_M)$ defined above, the derivative of $S$ with respect to $c$ is

$$\frac{dS}{dc} = \sqrt{\text{var}(V_M)} \equiv \sigma(V_M),$$

which is of course positive. So, in summary, a more risk averse market has less money invested but that money left in the market is required to earn a relatively higher risk premium (expected return minus risk-free rate), i.e. a higher Sharpe ratio or risk premium *relative* to standard deviation.

The usual diagram of the efficient frontier and the tangent coming off it (the CML) depict the CAPM in its "returns" equilibrium, but do not give a full picture of market risk aversion because there is no information in this famous image about the *absolute* prices of the assets that underlie their expected returns and returns variance. The dollar amount held in the market is obviously part of how it shows its appetite for risk. The efficient frontier diagram represents an expected utility maximizing market but does not give any hint of the total market value or total utility expected from that market.

## 5.1 Market Risk Tautology

Since market risk is partly diversifiable risk, the terms "undiversifiable risk" and "market risk" (also called "priced risk") are not synonyms. And since all market risk is "priced", the market return for risk is partly a return for taking diversifiable risk. Put another way, assets are discounted partly because they add diversifiable risk to the market portfolio, contrary to the rule that all diversifiable risk goes unrewarded.

Undiversifiable risk and "market" risk are easily distinguished. Undiversifiable risk can be identified from a payoff covariance matrix without knowing the equilibrium "market portfolio", but "market" risk cannot be identified from that matrix until the market portfolio is known (i.e. until all assets are priced).

If the market wanted to diversify more diversifiable risk, it would re-price assets so as implicitly to shift the efficient frontier, typically further to the left or upwards, thus leaving the CML steeper and the Sharpe ratio higher (assuming no change in exogenous $R_f$). The statement that "only market risk is priced" is tautologous, because "market risk" is just the name given to that risk which the market chooses to carry, some of which is diversifiable, and all of which is "worth the risk" (under a given level of risk aversion, represented by $c$).

Like so much in the CAPM, the notion of "systematic" or "market" risk is logically circular, by design. Assets are priced relative to a "market portfolio" that is itself unknown until those prices are known. The genius of the CAPM is that it solves the equilibrium problem of finding asset prices that satisfy a condition that is specified circularly as a function of those unknown prices. The effect is that when all assets in the market have their CAPM prices, the best portfolio of risky assets to hold is the one with portfolio weights exactly proportional to those very same prices.

# 6 Link Between Fama and Sharpe Ratios

If we define the *ex ante* Sharpe ratio $S$ of the market conventionally as in (9), following Sharpe (1964; 1994), some simple algebra reveals the relationship between the market's Fama ratio $\digamma_M = \text{var}(V_M)/E[V_M]$ and its Sharpe ratio. One way to write this relationship is as follows

$$P_M = \frac{S^2(1 - c\digamma_M)}{\digamma_M R_f \, c^2}. \tag{10}$$

In this equation, the equilibrium CAPM price of the market is written in terms of four market parameters, the Sharpe ratio, the Fama ratio, the risk free rate and the market risk aversion $c$.

Alternatively, writing the Sharpe ratio as a function of the Fama ratio

$$S = c\sqrt{E[V_M]\,\digamma_M}. \tag{11}$$

Most remarkably of all, the Sharpe ratio can be written as

$$S = c\sqrt{\text{var}(V_M)} \equiv c\,\sigma(V_M). \tag{12}$$

Like the equation (7) for beta in terms of payoff parameters, this strikingly simple relationship uncovers the economic "source" of the market's Sharpe ratio. Specifically, the primitive quantities that drive it are just the market's payoff variance or standard deviation and the market's aversion to payoff variance, $c$.

Surprisingly and defying intuition, not even $R_f$ affects the market's equilibrium Sharp ratio, despite the fact that in the usual returns expression of the Sharpe ratio, my (9), $R_f$ appears explicitly. Instead we find that, under the surface, in the inner workings of the CAPM equilibrium mechanism, the value of $R_f$ makes no difference to $S$. Rather, $R_f$ can change and the slope of the CML does not, which implies that there must be a compensating vertical shift in the tangency point on the efficient frontier. See the numerical illustration and plots below.

Equations (10)-(12) are interesting for what they show about hidden inner features of CAPM, and like (4), (5) and (7) they appear to be new and to add usefully to the mathematical insights from CAPM.

One important qualification is that the Fama ratio is exogenous, whereas the Sharpe ratio is endogenous. The Fama ratio is fully implied by the *ex ante* joint payoff distribution, whereas the Sharpe ratio is the gradient of the capital market line, and appears only after the CAPM equilibrium asset prices are known. Put another way, the Fama ratio of the market $\text{var}(V_M)/E[V_M]$ is given by the exogenous payoff distribution, whereas the market's Sharpe ratio is determined by the market itself, specifically by its variance aversion $c$.

I have referred to $c$ as the market's "variance aversion", as seems plainly true from the pricing equation (1). However, looking harder at (1), it is meaningful to think of $c$ as an exchange rate between payoff variance and payoff mean, since the mean payoff has a "one" in front of it in the CAPM price equation (1). This helps to explain why the Sharpe ratio, which captures a comparison between two parameters, returns mean and returns standard deviation, appears in (12) to be influenced only by the payoff variance and the "penalty for payoff variance", $c$. In fact, $c$ is not merely a penalty for variance or measure of aversion to variability; it is a parameter specifying or implying the market's required tradeoff between payoff variability and mean, or risk and return.

# 7 Numerical Example

For illustration of the above results, take the case of a hypothetical two-asset market with exogenous asset mean payoffs $E[V_1] = 150$, $E[V_2] = 200$, payoff variances $\text{var}(V_1) = 300$, $\text{var}(V_2) = 500$, and payoff correlation coefficient $\rho = -0.25$. The payoff covariance is therefore

$$\text{cov}(V_1, V_2) = \rho\sqrt{\text{var}(V_1)\,\text{var}(V_2)} = -0.25\sqrt{300 \times 500} = -96.83,$$

the market payoff variance is

$$\mathrm{var}(V_M) = \mathrm{var}(V_1) + \mathrm{var}(V_2) + 2\mathrm{cov}(V_1, V_2) = 606.351,$$

and the market Fama ratio is $\mathcal{F}_M = \mathrm{var}(V_M)/E[V_M] = 606.351/350 = 1.732$.

The two asset prices $P_1$ and $P_2$ are given by (1) under an assumed value of the market risk aversion $c$. The return on asset $j$ ($j = 1, 2$) is $r_j = V_j/P_j - 1$. Using the usual equations for the mean return and standard deviation of return for a weighted portfolio, the efficient frontier and CML arise endogenously from (i) the given value of $c$, and (ii) the exogenous payoff distribution. See examples in Figure 1.

Four sets of results are provided in Table 1 and illustrated in Figure 1. These assume $R_f = (1 + r_f) = 1.05$ and $c = 0.01$, 0.05, 0.075, and 0.1.

Table 1

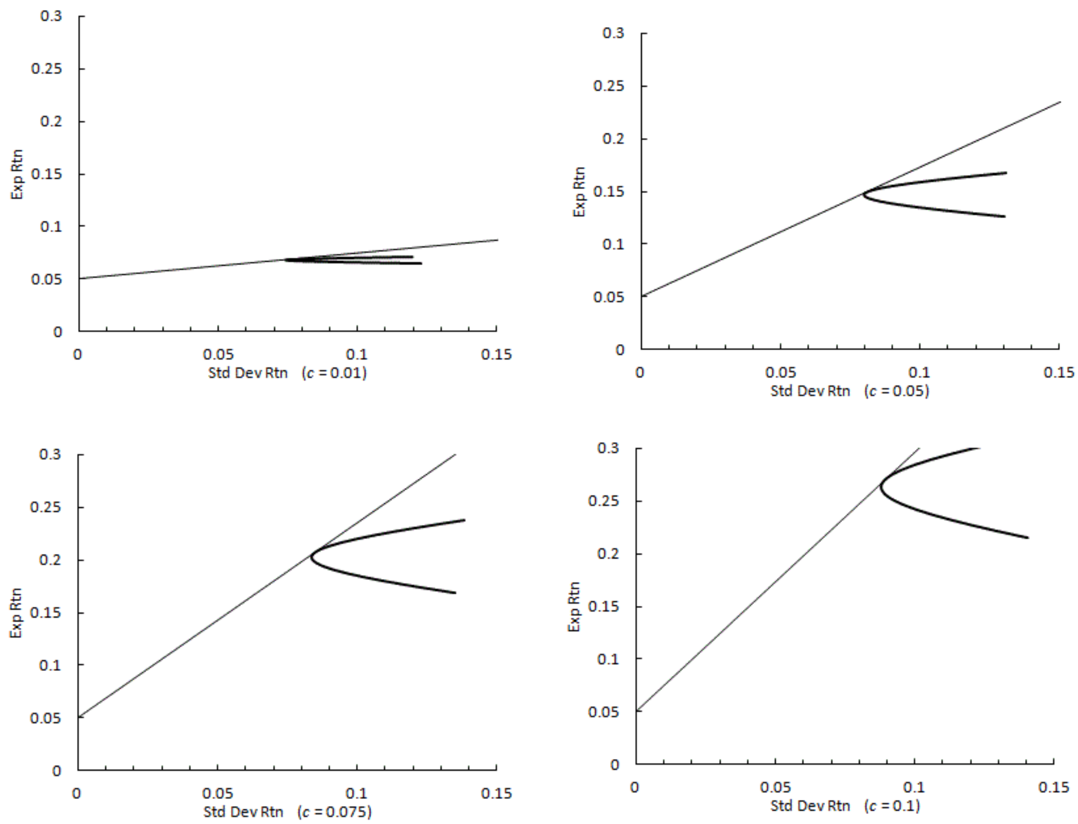
| $c$ | $E[r_M]$ | $\sigma(r_M)$ | $P_M(= P_1 + P_2)$ | Sharpe Ratio | $P_1$ | $P_2$ |
|---|---|---|---|---|---|---|
| 0.01 | 0.069 | 0.075 | 327.56 | 0.246 | 140.922 | 186.636 |
| 0.05 | 0.150 | 0.081 | 304.46 | 1.230 | 133.182 | 171.277 |
| 0.075 | 0.207 | 0.085 | 290.02 | 1.847 | 128.345 | 161.678 |
| 0.1 | 0.270 | 0.089 | 275.59 | 2.460 | 123.507 | 152.079 |

There is enough in the given calculations to confirm all of the relationships above. Note that the value of the market's Sharpe ratio can be cross-checked, which helps verify (12). Using the values in the table we can simply plug into the usual definition of the Sharpe ratio, (9). Alternatively, using the value of $c$ in the table, $S$ can be found by the remarkably primitive relationship (12) that involves just $c$ and $\mathrm{var}(V_M)$.

Figure 1 in effect shows four possible markets for the same two assets, each different in only its risk aversion, $c$. With any change in $c$, the market arrives at a new efficient frontier, and hence a new tangency portfolio and CML. Higher values of $c$ coincide with a lower total market value $P_M$ and a steeper CML (and thus higher Sharpe ratio). Lower $P_M$ tells us that a more risk averse market will pay less for the same assets with the same total expected payoff $E[V_M]$. Note how both the shape and location of the efficient frontier shift with changes in $c$. Higher $c$ is associated with a steeper CML and larger expected return $E[r_M]$, along with higher market return standard deviation $\sigma(r_M)$, illustrating what was proved above.

Figure 1

Efficient Frontier and CML for $c = 0.01, 0.05, 0.075, 0.1$ $(r_f = 0.05)$




This illustration helps explain how the CAPM works. The CAPM is understood as an equilibrium mechanism that finds its own efficient frontier and CML endogenously, in response to an exogenous joint payoff distribution, a given value of $c$, and a fixed risk-free rate.[8] The CAPM tangency portfolio has weights $P_j / \sum P_j$, which implies that at CAPM prices the optimal portfolio of risky assets under CAPM has weights proportional to those very same prices. This logically circular condition defines the CAPM equilibrium and is the essence of its intuitive appeal.

A more common understanding, perhaps because of the dominance in heavily CAPM fields (accounting and finance) of empirical work, is that asset returns and the efficient frontier are exogenous and simply observed. On this way of thinking,

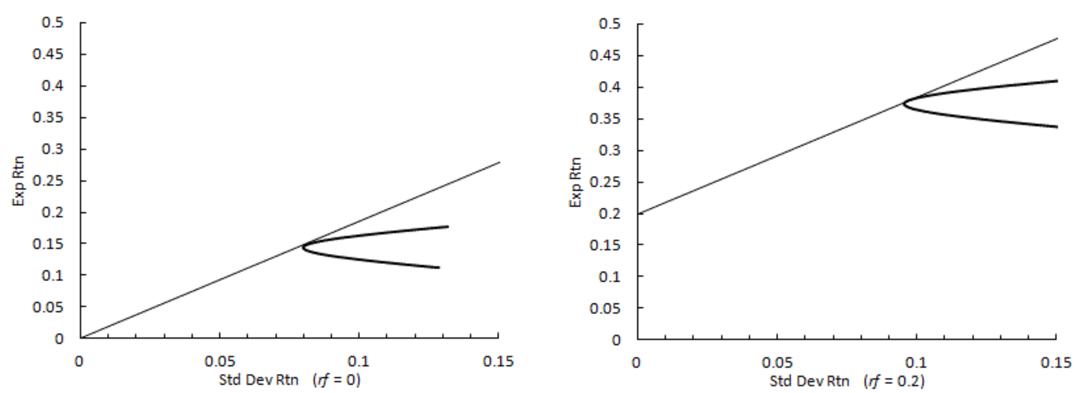
[8] The spreadsheet that shows this happening, and allows the exogenous inputs to be varied, is available from the author on request.

which is typical in the classroom, the CML is just the steepest line that touches the stationary exogenous efficient frontier. More correctly, the CML and efficient frontier should be seen as endogenous "variables" - they both move with $c$, and importantly they move as one.

The last step in this illustration is to show the curious fact of the equilibrium value of the Sharpe ratio being unaffected by the (exogenous) risk-free rate. The two plots in Figure 2 show results for the same two-asset market and payoff parameters as above, and assuming for illustration $c = 0.075$. The only difference is that we let $r_f$ take two new values, here $r_f = 0$ and $r_f = 0.20$. The Sharpe ratio or slope of CML remains in each case at 1.847, just as it was in the case above of $r_f = 0.05$ (with $c = 0.075$).

Figure 2

Efficient Frontier and CML for $r_f = 0$ and $r_f = 0.20$ $(c = 0.075)$




## 7.1 Previous Literature

The results above do not appear in textbooks, and beg the question of where they do occur. Their nearest precedent in existing finance literature is the under-appreciated discussion by Fama (1977) on how to use the CAPM as a strategic decision tool. In that early paper in the history of CAPM, Fama (1977, p.7) observed without further detail that "the relative risk measure $\beta_i$ is directly related to $\text{cov}(V_{it}, V_{Mt})/E(V_{it})$" (where in his notation $i$ is the firm and $t$ is the time).[9] Furthermore, Fama concluded that

[9] A similar but not identical ratio appears as the characertization of CAPM asset risk in the very early paper by Stapleton (1971, pp.109-110).

> ...this ratio of covariance to expected value [payoff] is the source of differences in the values of E($R_{it}$) for different firms. (Fama 1977, p.7)

Despite this lengthy paper being widely cited, there has been little if any ensuing recognition of the point in the quote and of "Fama's ratio" as the driver of expected return.[10]

The theoretical paper that is closest to Fama (1977) appears to be Hull (1986). Hull's proof of how there is more driving the discount rate than payoff "risk" or covariance alone is very similar to Fama's. He shows that the relationship between $E[R_j]$ and $\text{cov}(V_j, V_M)$ is non-linear through the effect of the payoff mean, $E[V_j]$. His proof starts by writing the CAPM expected return unusually as

$$E[r_j] = r_f + \frac{\lambda \text{cov}(Y_j, r_M) R_f}{1 - \lambda \text{cov}(Y_j, r_M)},$$

where $\lambda = (E[r_M - r_f])/\text{var}(r_M)$ is the usual CAPM "price of risk", $Y_j \equiv V_j/E[V_j]$, and $\text{cov}(Y_j, r_M)$ is a newly defined characteristic summarizing all that is relevant about the random payoff or firm under CAPM.

This result is easily reconciled with Fama's result as follows:

$$\text{cov}(Y_j, r_M) = P_M \ \text{cov}(\frac{V_j}{E[V_j]}, V_M) = P_M \ \frac{\text{cov}(V_j, V_M)}{E[V_j]} = P_M \digamma_j.$$

So, Hull's $\text{cov}(Y_j, r_M)$, like the much better known CAPM "beta" $\beta_j$, is a sufficient but not minimal sufficient index of the asset's risk class under CAPM. Treating $P_M$ as a market parameter too large to be influenced by a single risky project, the only facet of that project required to define its risk class is its Fama's ratio $\digamma_j$.

[10] This intepretation of CAPM is again in print, if not in mind, after Lambert, Leuz and Verrecchia (2007) re-found the Fama (1977) CAPM comparative statics. After showing the effect of the mean on the CAPM cost of capital, Lambert et al. (2007) emphasized in their analysis the better known and conventional point of how information drives the cost of capital through the market's subjective re-assessment of the covariance matrix of the joint payoff distribution. Fama's initial revelation of the role of the mean came about essentially because he looked at the CAPM in its payoffs or "values" form, rather than its usual returns form. Coles and Loewenstein (1988) used the payoffs form of CAPM to uncover a fundamental revision to how we understand the effects of parameter uncertainty in portfolio theory. The payoffs form of CAPM involves subjective probability distributions and Bayesian updating, whereas the returns form of CAPM can be viewed more empirically as a "regression equation", testable with historical data. There is an element therefore of Bayesian subjectivity versus classical frequentist methodology in the two different CAPM (re)expressions. The more *ex ante* subjectivist mindset, and its inevitable Bayesianism, is less familiar, as pointed to by Lewellen and Shanken (1992).

Hull (1986) warned that it is tempting yet wrong to claim that the CAPM cost of capital is determined by only firm or project "risk", in the sense of statistical variation. This mistake occurs time and again in the literature, and such claims, expressed in any number of ways, are the bread and butter of finance classes and business practice.

In a few notable cases, readers are misled only if they overlook the fineprint. In an important early exposition on CAPM as a capital budgeting tool, Turnbull (1977, p.1134) gave the following warning:

> Use of a single capitalization rate implicitly assumes that all cash flows are equally risky...

This familiar sounding statement might seem clear enough of itself, but later in the same paper its message is given much sharper meaning. Rather than leaving the notion of "risk" as a primitive term, undefined and up to the reader to interpret intuitively, the vital qualification is that payoffs are in the same risk class, or have the same "systematic risk", only if they are strictly proportional to one another

> ...the systematic risk of the projects will differ from that of the firm, unless the cash flows of the projects are strictly proportional to those of the firm. (Turnbull 1977, p.1138)

This requirement of proportional payoffs implies that they must have the same Fama ratio. Specifically, a random payoff of amount $hx$, where $h > 0$ is a constant, has Fama ratio

$$\frac{h\ \mathrm{cov}(x, M)}{h\ E[x]} = \frac{\mathrm{cov}(x, M)}{E[x]} = \digamma_x.$$

The only difference therefore between Fama (1977) and Turnbull (1977) is that Fama's requirement is more general. Two random cash payoffs, say $x$ and $y$, can have the same $\digamma$ without being proportional. Turnbull's requirement of proportionality is sufficient, but not necessary, for the two payoffs to be in the same CAPM risk class. This distinction highlights the irreducible quality of the Fama ratio. There is no criterion that defines CAPM risk class either more generally or more efficiently.

Proportional payoffs, like $x$ and $hx$, are perfectly correlated. Contrary to some intuition, cash payoffs can be perfectly positively correlated but not have the same

CAPM discount rate. Take payoffs $x$ and $y = a + bx$, where $a$ and $b > 0$ are constants ($a$ can be negative, like fixed costs). The Fama ratio of $y$ is

$$\frac{\text{cov}(a + bx)}{E[a + bx]} = \frac{b\,\text{cov}(x)}{a + bE[x]},$$

which equals the Fama ratio of $x$ only if $a = 0$, which is the special case where the two payoffs are proportional. So perfect linear correlation guarantees that the two payoffs in question have *different* CAPM costs of capital in all but the special case where their payoffs are strictly proportional ($a = 0$).

An important point here is again not to confuse payoffs with returns. If two assets have perfectly positively correlated returns, they have the same beta but if they have perfectly positively correlated payoffs, they have the same beta only if they have strictly proportional payoffs.

# 8 Do Unpriced Risks Exist?

Firm-centric activities are seen as unpriced risks for the reason that their payoffs have virtually zero, correlation with the market. The point which seems generally unknown is that despite having very low covariance with other assets or market activities, and thus adding negligibly to the market payoff variance, the activity in question can still have a high Fama ratio and hence a high returns beta.

To make things interesting, think of the real life search in the Caribbean Sea for the Spanish galleon San Jose and its lost treasure. A broadly realistic *ex ante* CAPM analysis of the San Jose venture goes as follows. Imagine that we have enough evidence *ex ante* to know that if we find the ship its treasure will be worth \$5 billion (the vessel was indeed found in 2014 with gold and other treasure worth \$5-15 billion). Also let us say that we give ourselves just a $p = 0.05$ probability of success. Failure has a 0.95 probability and will leave zero payoff. Thus, $V_j$ will be either 0 or $5 \times 10^9$, and $E[V_j] = 0.05(5 \times 10^9) = 2.5 \times 10^8$ and $\text{var}(V_j) = E[(V_M - E[V_M])^2] = 1.1875 \times 10^{18}$.

The next step is to find a discount rate. A successful search relies entirely or almost entirely on unique "project-specific" factors (e.g. the accuracy of the historical research pointing to the search location). It is rational nonetheless to see success as having some small positive correlation with the market, since many underlying factors (like the solvency of contractors or backers, the sale proceeds from the treasure, etc.)

are market related. It is shown below that the venture's correlation with the market can be near zero or very low while at the same time beta is large.

To introduce market correlation into the analysis, take the market as either good or bad and let the probability of success be $p_G$ under "good" and $p_B$ under "bad", where $p_G > p_B$. To make the scale of the market realistic in relation to the project scale, take $V_M$ as the value of the US stock market, and say that it is currently worth \$20 trillion, and has two possible future values, these being $V_M = 23 \times 10^{12}$ in a "good" market and $V_M = 19 \times 10^{12}$ in a "bad" market. These are seen as equally likely. The expected market return is then $E[R_M] = E[V_M]/P_M = 1.05$ and the market payoff variance is $E[(V_M - E[V_M])^2] = 4 \times 10^{24}$.

For simplicity, and because the issue is the risk premium, the risk-free return is taken as zero, i.e. $r_f = 0$ $(R_f = 1)$.

Based on these assumptions, the implied market variance aversion parameter is $c = (E[V_M] - P_M R_f) / \mathrm{var}(V_M) = 2.5 \times 10^{-13}$. This is quite realistic (see the empirical estimate of $c$ later in the paper).

To apply CAPM correctly, the discount rate $E[R_j] = E[V_j]/P_j$ is deduced from the pricing form of CAPM

$$P_j = E[V_j] - \frac{\mathrm{cov}(V_j, V_M)}{\mathrm{var}(V_M)}(E[V_M] - P_M R_f). \qquad (R_f = 1)$$

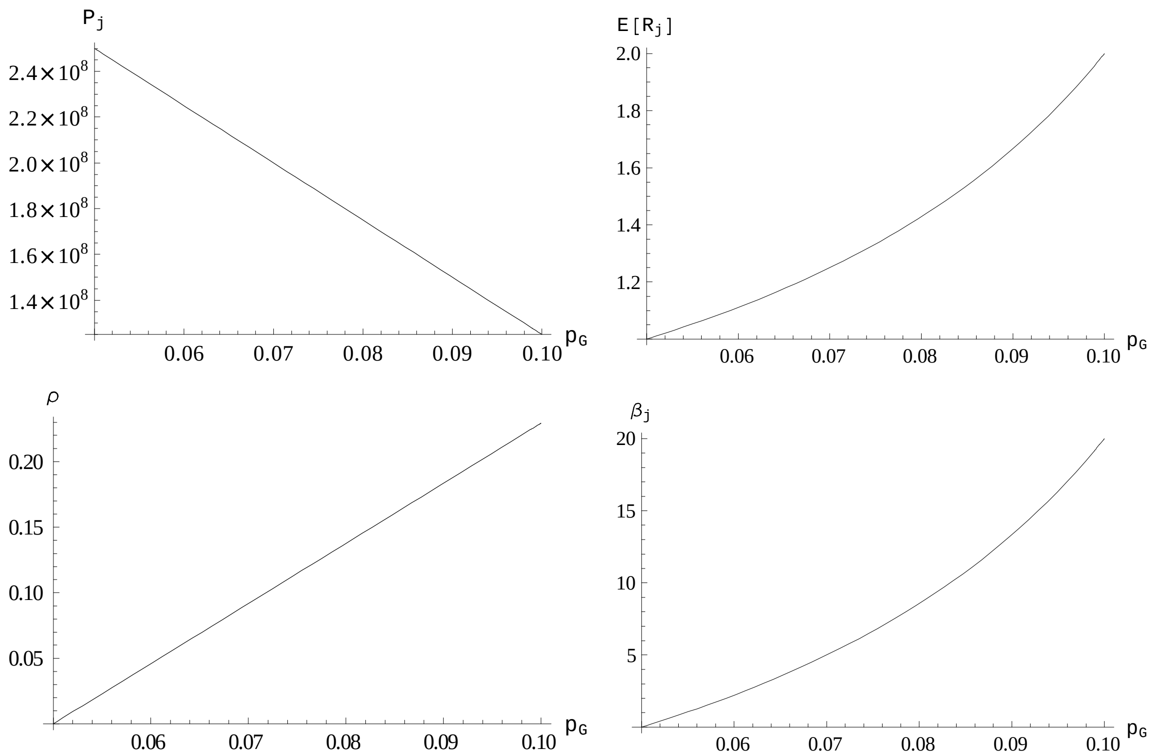
Figure 3 shows results for $p = 0.05$ and $p < p_G < 0.1$, remembering that the assumptions above require $0.5p_G + 0.5p_B = p = 0.05$. Plots are provided for the *ex ante* venture value $P_j$, the CAPM implied discount factor $E[R_j] \equiv E[1+r_j] = E[V_j]/P_j$, the correlation coefficient $\rho = \mathrm{cov}(V_j, V_M)/\sqrt{\mathrm{var}(V_j)\mathrm{var}(V_M)}$ between venture and market payoffs, and the CAPM beta of the venture, $\beta_j$. The *ex ante* project beta is easily calculated from (3), (4) or (7).

This case is revealing for how it contradicts the staple textbook principle that "idiosyncratic" projects have low beta and the market does not demand a high risk premium on projects that have low correlation with the market.

The calculations show that it can take only very small positive correlation between the risky project and the market for its CAPM beta and discount rate to become large. For example, if we suppose that $p_G = 0.06$, implying that $p_B = 0.04$ (thus holding $p = 0.05$ as assumed), then the state of the market is perceived as almost irrelevant to the probability of success, since that probability remains little different no matter what the market state. On this *ex ante* assessment of the venture's chances, the correlation coefficient between venture and market is only 0.046, and yet the CAPM beta is 2.222, leaving a required return of 11.11% ($E[R_j] = 1.1111$) (i.e. more than twice the market average).

Figure 3

Plots of $P_j$, $E[R_j]$, $\rho$, and $\beta_j$ (for $0 < p_G < 0.1$)




Despite its minimal dependence on the market, the venture is by no means a "zero beta" activity. Furthermore, if that dependence (governed by $p_G$) increases even just a little, beta rises extremely quickly, as is shown in the plot of $\beta_j$ against $p_G$. For instance, with $p_G = 0.065$, $\beta_j = 3.53$, even though the correlation coefficient is still only $\rho = 0.069$, as can be seen in Figure 1. Similarly, holding the mean payoff constant, a correlation of about 0.25 is enough to produce a beta of 20.

The general rule that sums up this analysis is that projects with low chance of success, or low mean payoff, can have very high beta even when they have very low correlation with the market. These projects demand a high CAPM discount rate, much as pre-CAPM financial intuition would suggest. They are obviously "risky", but are not risky in the sense of having high payoff correlation with the market, and yet they have a high CAPM beta and hence discount rate. This is clear only once we understand that their high discount rate traces to their very low mean payoff relative to their low covariance with the market (i.e. to their high Fama ratio $\digamma_j$).

The lessons here are twofold. First, no actual business venture has exactly zero correlation with the market and thus zero $\digamma_j$. In CAPM decision theory, covariance with the market is a forward-looking subjective perception of the association between the payoff and the market, and there are always dependencies by which an apparently "firm-specific" cash receipt hinges on market variables. Even in sports betting, which is sometimes painted as a "zero beta" activity, the gambler depends on the bookmaker making good on winning bets, which itself depends to some extent on market conditions. Second, and relevant to all actual business decision making, even ventures that seem highly idiosyncratic can have a very high forward-looking beta estimate. The moral is not to be fooled into thinking that the apparent firm-specific uniqueness or "idiosyncrasy" of a project, especially a "long shot" with relatively low $E[V_j]$, excuses that venture from a possibly high CAPM discount rate.

## 8.1 How This Counter-Example Works

The numerical example above was set up to show how highly idiosyncratic risks can have (very) high beta. The insight that leads to it is simple. Differentiating (2) with respect to the covariance gives

$$\begin{aligned} \frac{d\,(E[R_j])}{d\,(\mathrm{cov}(V_j,V_M))} &= \frac{d\,(E[V_j]R_f \,/\, [E[V_j]-c\,\mathrm{cov}(V_j,V_M)])}{d\,(\mathrm{cov}(V_j,V_M))} \\ &= \frac{c\,R_f\,E(V_j)}{(E[V_j]-c\,\mathrm{cov}(V_j,V_M))^2}. \end{aligned}$$

So the CAPM cost of capital $E[R_j]$ reacts very strongly to small changes in the payoff covariance when $E[V_j]$ is not too much greater than $c\,\mathrm{cov}(V_j,V_M)$, assuming the usual case where both mean and covariance are positive. If $\mathrm{cov}(V_j,V_M)$ is exactly zero, which is mostly unrealistic, the discount rate is $R_f$, but any small positive covariance can be enough, when the asset's mean payoff is low enough, to send its discount rate into rapid incline (as in Figure 3).

Note that the San Jose example is typical of many risky ventures, and still has a positive CAPM price, or potentially positive NPV, despite its low probability of success. Its cost of capital is higher than might be concluded from the fact that it is highly idiosyncratic, firm-specific or unique, which are all terms used in finance classes to describe ventures that present nearly entirely "diversifiable" or "unsystematic" risk rather than "systematic" or "priced" risk. Interestingly, this CAPM insight takes us closer to a pre-CAPM ethos, where obviously risky ventures like the San Jose would not be excused from high discount rates on account of their being so "unique" of "firm-specific".

## 8.2 Market Can't Absorb All "Unique" Risks

If enough firms take enough unique but highly "risky" investments characterized by low mean payoffs relative to their (low) payoff covariances with other assets or firms, the market's own Fama ratio $\digamma_M$ must increase. By (6) this will add to the market risk premium.

This argument contradicts the CAPM myth that markets can absorb any number of idiosyncratic bets taken by firms, no matter how risky these are in the sense of their probabilities of failure, without any individual firm, or the market as a whole, paying any price in terms of higher discount rates. The best that can be said is that a "more diversified" market, in the sense of a market with lower Fama ratio, will impose a lower price of risk across firms in aggregate.

The converse argument is also unusual. If firms take on more low covariance-to-mean investments, $\digamma_M$ will decrease and with it the market risk premium. The probability distribution of the aggregate market payoff $V_M$ will shift right but with little change in market payoff variance, leaving a lower $\digamma_M$ and hence lower $E[r_M]$. Such activities are of course unlikely to exist in any great supply.

# 9 A CAPM Paradox

The following argument shows by simple algebra that an asset with precisely zero payoff covariance with every other asset in the market can theoretically have a high discount rate under CAPM.

An asset $j$ with zero payoff covariance with every other asset has market covariance $\text{cov}(V_j, V_M) = \sum_k \text{cov}(V_j, V_k)$ equal to its variance $\text{cov}(V_j, V_j) \equiv \text{var}(V_j)$, since $\text{cov}(V_j, V_k) = 0$ for all $k \neq j$. Now suppose that its payoff variance is increasing relative to its mean, either because of a higher variance or lower mean, in which case its CAPM price $P_j = (E[V_j] - c\,\mathrm{var}(V_j))\,/R_f$ tends towards zero. Assuming that its price $P_j$ remains positive but approaches zero, then its Fama ratio

$$\digamma_j = \frac{\mathrm{var}(V_j)}{E[V_j]} < \frac{1}{c} \to \frac{1}{c},$$

and hence its CAPM discount factor, from (5),

$$E[R_j] = R_f\ [1 - c\ \digamma_j]^{-1} \to \frac{R_f}{0}$$

must increase with increasing $\mathrm{cov}(V_j, V_M)$ and must ultimately approach infinity as $P_j \to 0$.

Thus, an asset with positive CAPM price and zero correlation with every other existing asset can have a discount rate under CAPM approaching infinity. This result is paradoxical under CAPM because the asset described, being uncorrelated with every other asset, could not be more "idiosyncratic". Such an asset attracts a high discount rate, yet there is no material effect on the aggregate market average discount rate for the reason that its CAPM price is so low that on a price-weighted basis it has no influence on the market average. So, just as in the example above, a "unique" or entirely idiosyncratic asset does not add to the market risk premium, yet still draws a heavy risk premium or discount rate at the individual asset level.

## 9.1 Empirical Approximation of $c$

The analysis above proves that any asset with $P_j \to 0$ has necessarily $E[R_j] \to \infty$. This result is "asymptotic" in that the rapid increase in the discount rate occurs only as $P_j$ gets very close to zero.

The practical question of whether in reality an asset can have a payoff variance that is so large relative to its mean that its $\digamma_j$ is high enough to attract a high CAPM discount rate even when it has zero correlation with all other assets. From (5) we have the CAPM discount factor $E[R_j] = R_f\ [1 - c\ \digamma_j]^{-1}$. Hence, to reveal how large $\digamma_j$ must be to make $E[R_j]$ much greater than $R_f$, we need an empirical estimate of $c$ from the stock market.

The parameter $c$ is revealed in the market by the equilibrium "price of payoff variance"

$$\begin{aligned} c &= \frac{(E[V_M] - P_M R_f)}{\text{var}(V_M)} \\ &= \frac{P_M\, E[r_M - r_f]}{P_M^2\, \text{var}(r_M)} \\ &= \frac{1}{P_M} \frac{E[r_M - r_f]}{\text{var}(r_M)}. \end{aligned}$$

An empirical estimate of $c$ is found by taking estimates of the market risk premium and market return variance from historical data. A simple approach gives

$$\widehat{c} = \frac{E[\widehat{r_M - r_f}]}{\widehat{\text{var}(r_M)}} \frac{1}{P_M} = 2.35 \times \frac{1}{26 \times 10^{12}} = 9.02 \times 10^{-14},$$

where $E[\widehat{r_M - r_f}] = 0.0631$ is the long run average market risk premium on US stocks (1960-2015), $\widehat{\text{var}(r_M)} = 0.0269$ is the variance of the market return over the same data series, and $P_M = \$26$ trillion is the current total price of stocks in the US (approximated by \$19 trillion of the NYSE and \$7 trillion on Nasdaq).

This approach gives a plausible approximation of $c$. The estimates of long run market risk premium and market return variance are obtained from the annual data series on the data site provided by Aswath Damodaran [11]

Given $\widehat{c}$, it is an easy task to work back using (5) to the magnitude of Fama ratio $F_j$ required in reality to warrant a given cost of capital. It is quickly seen that an asset with zero correlation with every other asset will not in general have $F_j$ anywhere near large enough to attract a large CAPM discount rate (5). So the paradox above is a limiting case. It does however apply to "penny dreadful" assets with price $P_j \to 0$. That follows mathematically from CAPM, as shown above.

# 10 Diversifiable Risk Priced at the Firm Level

Risk that is perfectly diversified and adds nothing to market portfolio variance, can affect the individual firm's cost of capital. In more general terms, such risks affect

[11] The annual market returns $r_M$ are from the S&P 500 over 1960-2015 (inclusive) and the premiums are found from the simultaneous 3 month T-bill rates. See the data at http://www.stern.nyu.edu/~adamodar/pc/datasets/histretSP.xls

the firm's Fama ratio and cost of capital but not the market Fama ratio and risk premium.

Suppose that firm $A$ has cash payoff $V_A = a + x$ and firm $B$ has payoff $V_B = b - x$, where $a, b$ and $x$ are random. The two firms' costs of capital will be affected by the mean and covariance parameters of $x$, but the market return will not. The two firms' Fama ratios are

$$F_A = \frac{\text{cov}(a + x, V_M)}{E[a + x]} = \frac{\text{cov}(a, V_M) + \text{cov}(x, V_M)}{E[a] + E[x]}$$

and

$$F_B = \frac{\text{cov}(b - x, V_M)}{E[b - x]} = \frac{\text{cov}(b, V_M) - \text{cov}(x, V_M)}{E[b] - E[x]}$$

which can be greatly altered by $E[x]$ and $\text{cov}(x, V_M)$. But the effect of $x$ cancels out in the market payoff, leaving both the mean market payoff $E[V_M]$ and variance $\text{var}[V_M]$ unchanged, whatever the statistical properties of $x$.

This simple case shows that there is a kind of diversifiable risk that is priced by the CAPM at the firm level and yet unpriced at the market level. The firm level effects cancel themselves out at in the market aggregate. This counter-example contradicts the much touted CAPM edict that "one gets no compensation or risk adjustment for holding *diversifiable* risk".

A more standard looking numerical example is set out in the Appendix.

# 11 Information Risk Priced

There are two views on "information risk" (also called parameter or estimation risk). One position, tested empirically in the accounting literature (e.g. Francis, LaFond, Olsson and Schipper, 2005), regards each firm's individual disclosure quality as firm-specific and hence as "diversifiable", implying that the firm pays no price for emitting poor quality information. The rival conclusion, standing on the parameter risk literature in Bayesian portfolio theory (e.g. Brown, 1979; Klein and Bawa 1976; 1977; Coles and Loewenstein, 1995; Lambert et al. 2007), is that better information quality is reflected in generally greater certainty about future cash payoffs, implying therefore a generally lower assessment of the firm's beta, and thus a lower discount rate.

To decide between these rival intuitions, a helpful shortcut is to think of how accounting for information risk might alter the firm's Fama ratio (which is forward-looking and hence subjective). This requires an explicitly Bayesian model, since the two (unknown) parameters in Fama's ratio are the mean and covariance of the firm's payoff under the Bayesian predictive joint distribution of all uncertain asset payoffs in the market.

A strict Bayesian position is that all risk or uncertainty, no matter what its "type" or source, is assimilated within the conditional joint payoff distribution $f(\boldsymbol{V}|\Omega) \equiv f(V_1, V_2, ..., V_n|\Omega)$. In principle, $\Omega$ contains all existing knowledge, including anything that may help the decision maker interpret newly observed information. For instance, $\Omega$ might contain knowledge or evidence that a particular class of signals is highly reliable, or highly unreliable. The net effect is that the subjective probability distribution $f(\boldsymbol{V}|\Omega)$ conditions not only the particular signals that have been observed, but also on all information and beliefs about their reliability or about the mechanisms (e.g. experts, instruments, models or media) by which they were observed, and due to which they could possibly fail.

Suppose that a signal $\omega$ arises randomly from one of a number of possible sources $\theta$, each with its own perceived error characteristics, making its qualities random. The total available information $\Omega = \Omega_0 \cap \omega$ includes that signal $\omega$, along with the prior (pre-existing) information $\Omega_0$. Background information, or the conjunction $\Omega$ of background information and signal $\omega$,[12] might suggest that $\omega$ was generated by a given source, parameterized by $\theta$, with probability $f(\theta|\boldsymbol{V}, \Omega)$. This distribution, representing the probability that signal $\omega$ came from source $\theta$, is conditioned on $\boldsymbol{V}$ to allow for the possibility that the underlying state of $\boldsymbol{V}$ affects the source or error properties of the signal (e.g. high earnings might encourage less manipulation of reported earnings).

The required posterior distribution is then

$$f(\boldsymbol{V}|\Omega) = f(\boldsymbol{V}|\Omega_0 \cap \omega) = \frac{f(\boldsymbol{V}|\Omega_0) f(\omega|\boldsymbol{V}, \Omega_0)}{f(\omega|\Omega_0)},$$

where

$$f(\omega|\boldsymbol{V}, \Omega_0) = \textstyle\int_\theta f(\theta|\boldsymbol{V}, \Omega_0) f(\omega|\boldsymbol{V}, \Omega_0, \theta),$$

[12] The signal realization can often tell much about its own unknown source.

and

$$f(\omega|\Omega_0) = \int_{\boldsymbol{V}} f(\boldsymbol{V}|\Omega_0) f(\omega|\boldsymbol{V}, \Omega_0).$$

In these calculations, $\theta$ is a "nuisance parameter" and is integrated (i.e. averaged) out in the standard way that Bayesian inference eliminates nuisance variables. Note also that the information risk or perceived "error properties" of each possible observer state $\theta$ are embedded within likelihood function $f(\omega|\boldsymbol{V}, \Omega_0, \theta)$. It can be seen therefore that the Bayesian posterior distribution incorporates perceived information risk, and hence the Fama ratio of the posterior payoff mean and covariance for a given firm's payoff $V_j$ will be influenced by the market's perception of the information risk surrounding signal $\omega$.

This conclusion is really only a broader version of the general argument in Bayesian portfolio theory (Rachev, Hsu, Bagasheva, and Fabozzi (2008)). It leads however to the possibility that allowing for uncertain signal quality might lead to a higher Fama ratio and discount rate, but might also bring a lower Fama ratio and lower cost of capital. The latter possibility is in fact obvious because a Bayesian revision of beliefs about payoff $V_j$, based on a reassessment of the available information, can lead to a higher (or lower) mean and to a lower (or higher) covariance. That is, $E[V_j|\Omega_0 \cap \omega]$ might be higher than $E[V_j|\Omega_0]$ and $\text{cov}(V_j, \sum V|\Omega_0 \cap \omega)$ might be lower than $\text{cov}(V_j, \sum V|\Omega_0)$.

Intuitively, these possibilities must exist and have many possible ways of occurring. For example, if we allow for the innate conservatism in reported accounting earnings, we might increase our estimate of the firm's future cash flow. And if we allow for analysts' forecasts of different firm's earnings being generally herded and often wrong in the same direction, e.g. biased upwards, then we might revise our assessments of the (positive) covariances of those firms' earnings downward.

The conclusion from this argument is that Bayesian revision of a firm's Fama ratio, to formally allow for information risk, will lead to a new forward-looking discount rate, and that rate will sometimes be lower than the previous cost of capital, which stood on a less conditioned posterior distribution.

Thus, information risk is naturally "priced", but, in very much the same way as setting insurance premiums, there is always the chance that the risk premium for a given firm might come down when the perceived qualities of information are allowed for. Insurance risk premia are a good analogy. If a medical test is found to be unreliable, people who tested "positive" for the risk factor in question, and were charged a higher life insurance premium, may now be seen as lower risk and allowed a lower health insurance premium. Note that the Bayesian outcome of uncertainty increasing on the arrival of better information, was first emphasized by Barry and Winkler (1976) and more recently by Lewellen and Shanken (2002).

## 12 Probability Mixture Assets

There is practically no risk or other consideration about the firm's fundamentals that goes unpriced, or should go unpriced, under CAPM. If the firm builds a new factory, or a new product, or a new product costing system, those choices will inevitably affect its payoff distribution in ways that change its Fama ratio and hence its forward-looking discount rate. There is, however, one novel source of random payoff variation that can be strictly unpriced under CAPM. This unpriced risk occurs in a special case within the class of assets that might be called "randomized securities" or "probability mixture assets".

Define an $\alpha$-mixture of assets $A$ and $B$ as an asset that yields payoff $V_\alpha$ given by

$$V_\alpha = \begin{cases} V_A & \text{if } \theta = 1 \\ V_B & \text{if } \theta = 0 \end{cases}$$

where $\theta \in \{0, 1\}$ is a Bernoulli random event with index $\alpha$.

A "mixture asset", or "compound lottery" as it would be called in utility theory, is thus a random payoff with a mixture distribution of two (or more) different probability distributions.[13] Baron (1977, p.1692) and Liu (2004, p.233) discuss how probability mixtures of different assets can occur in business. For instance, the cash payoff from a project might have different probability distributions depending on a random event such as the outcome of a court case or regulatory decision.

The mean payoff of the mixture asset is just the probability weighted mean payoff of the two underlying assets

$$E[V_\alpha] = \alpha\, E[V_A] + (1 - \alpha)\, E[V_B],$$

[13] Note that this very contract exists in gambling and is called a "mystery bet". The gambler bets a fixed sum on a horse or other prospect that is chosen at random before the event by the gambling agency's software.

and, by the covariance decomposition formula, the covariance of the mixture asset is

$$\text{cov}(V_\alpha, V_M) = E[\text{cov}(V_\alpha, V_M)|\theta] + \text{cov}(E[V_\alpha], E[V_M]|\theta).$$

Interestingly, if event $\theta$ is seen as pure chance and independent of the market, like a coin toss, then the second term in the covariance, $\text{cov}(E[V_\alpha], E[V_M]|\theta)$, equals zero, because $E[V_M]$ is a constant with respect to $\theta$ (i.e. $E[V_M]$ is unaffected by the chance outcome $\theta$).[14] In that case,

$$\text{cov}(V_\alpha, V_M) = \alpha\,\text{cov}(V_A, V_M) + (1-\alpha)\,\text{cov}(V_B, V_M).$$

Conveniently, therefore, both the mean and covariance of the $\alpha$-mixture are simple $\alpha$-weighted averages of the two underlying payoff means and covariances. The Fama ratio of the $\alpha$-mixture asset is then

$$\digamma_\alpha = \frac{\alpha\,\text{cov}(V_A, V_M) + (1-\alpha)\,\text{cov}(V_B, V_M)}{\alpha\,E[V_A] + (1-\alpha)\,E[V_B]}. \tag{13}$$

It follows from (13) that if the two underlying assets have the same Fama ratio, any $\alpha$-mixture ($\alpha \in [0,1]$) of those assets has the same Fama ratio and cost of capital. So if two ventures happen to have the same rational discount rate, no matter how different they look in economic fundamentals, then the discount rate applicable to a contract (*ex ante* commitment) to invest in whichever project is drawn randomly, by whatever chance mechanism is nominated, is the same rate.

It is also easy to see that when the two assets have the same means, $E[V_A] = E[V_B]$, regardless of their payoff variances, the Fama ratio of any $\alpha$-mixture asset of two assets is the $\alpha$-weighted average of the two Fama ratios. In effect this implies that the discount rate is close to a linear average also. From (5), the expected return from the $\alpha$-mixture asset is $E[R_\alpha] = R_f\,(1 - c\,\digamma_\alpha)^{-1}$, so $E[R_\alpha]$ is very close to linear in $\digamma_\alpha$ when $E[R_\alpha]$ is in the typical range around say 1.2 or not greatly higher than $R_f$.

In the general case, where $\digamma_A \neq \digamma_B$, the randomized asset's Fama ratio $\digamma_\alpha$ and its expected return $E[R_\alpha] \equiv f(\alpha) = R_f\,(1 - c\,\digamma_\alpha)^{-1}$ vary non-linearly with $\alpha$. When plotted against $\alpha$, $E[R_\alpha]$ lies on an arc, between $E[R_A]$ and $E[R_B]$. That arc $f(\alpha)$

[14] Note the usual assumption that the mean payoff of any single asset $j$, $E[V_j]$, is negligibly small relative to the aggregate market payoff $E[V_M] \equiv E\left[\sum V\right]$. Thus, variation in $E[V_j]$ is not large enough to bring non-negligible variation in $E[V_M]$.

can be either concave or convex, meaning that an $\alpha$-mixture asset can have a cost of capital which is either higher or lower than the $\alpha$-weighted average of the two assets' costs of capital. Letting $\mu_j = E[V_j]$ and $\sigma_j =$cov$(V_j, V_M)$ for notational brevity, the second derivative of this function with respect to $\alpha$ is

$$f''(\alpha) = \frac{-2cR_f\big[c(\sigma_A - \sigma_B) - (\mu_A - \mu_B)\big](\sigma_A\mu_B - \sigma_B\mu_A)}{\big[c\sigma_B - \mu_B + \alpha\big(c(\sigma_A - \sigma_B) - (\mu_A - \mu_B)\big)\big]^3} \tag{14}$$

which can be positive or negative. The sign of (14) is determined by the relative values of the means and covariances of $A$ and $B$ and is unchanged over $\alpha \in [0, 1]$. In the case of $F_A = F_B$ mentioned above, $f''(\alpha) = 0$. The other special case where $f''(\alpha) = 0$ and hence $f(\alpha) \equiv E[R_\alpha]$ is linear occurs when

$$\frac{E[V_A] - E[V_B]}{\text{cov}[V_A] - \text{cov}[V_B]} = c.$$

In these special cases, the expected return on the mixture asset is just the $\alpha$-weighted average of the two assets' expected returns. Circumstances where $f(\alpha)$ is upward convex, and hence the required return or discount rate on the mixture asset is less than the $\alpha$-weighted average of the two constituent discount rates, are easily found.

The fact that $f(\alpha)$ is sometimes convex suggests tantalizingly that there can be something to gain economically by mixing investments using a randomizer. The same possibility occurs in frequentist statistics where hypothesis tests with different pairs of Type I and Type II error frequencies can be mixed by a randomizer to achieve a better compromise between the two error rates than is available from either one of the two underlying tests (see Lehmann 1986, pp.539-542 on mixtures of experiments). Essentially, the randomizer fills in the missing test space between the available Type I and Type II error pairs. A similar frequentist technique holds here, because the randomizer fills in a set of mean-covariance pairs that might not exist naturally in the real world asset market.

The question is whether any of these possibilities might be optimal relative to the pure strategies of simply investing in A or simply investing in B. Put another way, if a firm has two different projects available to it, each with its own payoff mean and covariance pair, is it possible by choosing between these two projects randomly, by spinning a wheel, that a preferable mean-covariance pairing can be obtained?

Unfortunately, the answer under CAPM is negative. The reason for this is that the CAPM price of one of the two projects is necessarily higher than the price of any $\alpha$-mixture of the two.[15] To see this, note that although $F_{\alpha}$ and $E[r_{\alpha}]$ change non-linearly with $\alpha$, the CAPM price (1) of the asset $P_{\alpha}$ is a linear combination of $E[V_{\alpha}]$ and $\text{cov}(V_{\alpha}, V_M)$, and since these are both linear in $\alpha$, so is $P_{\alpha}$. Specifically, the CAPM price of an $\alpha$-mixture of two assets is the $\alpha$-weighted average of their two prices, so logically $P_{\alpha} < \max(P_A, P_B)$ for any $\alpha$ $(0 < \alpha < 1)$ [16] Note also that the CAPM price of a mixture of two assets with the same price is that same price, which means that an investor is indifferent between choosing between these subjectively and having the randomizer do it.

The formal concept of a "mixture asset" is abstract, however it might seem that every project is in essence a mixture asset. For example, the payoff distribution of a new venture might depend on a random variable like a political decision. Thinking this way, the predictive payoff distribution is a mixture of different possible distributions. Crucially, however, the pivotal events $\theta$ that decide the final distribution are rarely ever independent of the market (i.e. they are not pure chance).

Once we account for the fact that the probability $\alpha$ is affected by market conditions, the results above no longer hold. Instead, treatment of a project as a mixture distribution, or mixture of mixture distributions, with a set of different pivotal events $E$ all driven by the market to some small or large extent, is just another probability model by which to arrive at the Bayesian posterior predictive distribution for $V$.

Some Bayesian models can easily be rewritten so that the posterior distribution has the explicit structure of a mixture of posteriors. In all such models, the risks represented by the market related $\alpha$'s are priced in the sense that they naturally affect $E[R_{\alpha 1, \alpha 2, ..}]$.

[15] Another obvious point is that the project chosen by the randomizer might be known after the wheel is spun, so the applicable cost of capital will be the one conditioned on that project. The same argument is made in Bayesian statistics, where the conclusion is that the error frequencies of the randomized test should be conditioned on the test selected, rather than being taken from the long run average error frequencies of the mixture test (Berger and Wolpert 1984, pp.6-14). This is a large part of the dispute between classical (frequentist) and Bayesian (conditional) statistical philosophies. See also Lewellen and Shanken (2002). The project chosen by a randomizer is sometimes not known, as when a firm pre-commits to a project knowing (say) that a forthcoming independent "random" event (e.g. court case) will affect its payoff distribution.

[16] For discusssion on the behavior of mixture assets on the mean-variance plane, see Borch (1969; 1974), Baron (1977) and Johnstone and Lindley (2013).

## 13 Conclusion

When articulated fully, the CAPM implies that virtually every aspect of a risky firm or venture, including matters that seem totally unrelated to its risk, affect its cost of capital. There is essentially no such thing as an idiosyncratic "unpriced" risky activity, in the sense described in textbooks. Activities that seem to be unique or firm-specific can have high "beta" and hence high CAPM cost of capital. Even when a risky payoff has apparently negligible positive correlation with the market it can have high beta, depending on whether its payoff covariance is large relative to its payoff mean. The fundamental relationship between these parameters that determines an activity's CAPM cost of capital is just their simple ratio. This ratio is fairly called the "Fama Ratio" after Fama's (1977) explicit, yet never mentioned, proof of how it drives the CAPM price-implied cost of capital.

If we accept what Fama showed, and all of its implications set out above, the question begging is why finance teaching oversimplified its interpretation of the CAPM. Part of the answer is that there is a natural temptation to speak of the risks of asset returns and the risks of asset payoffs in the same breath, as if one simply mimics the other, and so we overlook the CAPM fact that the risk premium hinges on just the second moment of returns but on both the first and second moments of payoffs.

Remarkably, this fact seems little known. Instead, asset prices are treated as if they are exogenous constants, which brings on the oversimplification that returns are just payoffs divided by a constant. The fact is that prices are constants only after they are fixed endogenously. Hence, returns are endogenous when payoffs are exogenous.

Having adopted an over-simplified CAPM pedagogy, obscuring the equilibrium logic by which CAPM puts "prices" on uncertain payoffs, and having found both students and practitioners captivated by agreeable yet not strictly correct edicts like "diversifiable risk is not priced", the impetus to think about the CAPM more deeply was lost, leaving a flawed understanding in common business discourse and a list of insightful yet widely unknown CAPM corollaries. One aspect of the literature that stands out is the relative importance that seems to have been placed in the 1970s on interpreting the CAPM as a logical calculus of asset returns. This philosophical depth contrasts with how students often see the CAPM as merely an empirical relationship, or regression equation. That mindset leads some to reject the CAPM altogether because it is hard to test or does not test well empirically. If we take that perspective, we lose sight of the "logic of CAPM" and the economic insights it brings to asset valuation and portfolio optimization, without which finance theory would have far less to offer.

Reading the CAPM as an *ex ante* pricing and decision tool, rather than as an *ex post* regression equation, requires that asset returns are viewed as by-products of an endogenous equilibrium or pricing mechanism, rather than of exogenous "nature". This distinction is often blurred and the intuitive understanding of CAPM that comes out of an explicitly payoffs perspective is widely missing.[17] Cochrane (2001, p.43) noted that most finance models treat the returns process as "statistical", making economic fundamentals like changes in a firm's cost of capital difficult to explain other than by abstraction or "metaphor". A common metaphor is to say that "the firm's cost of capital increased because the new project added to the firm's beta". A fundamental economic explanation is something more like the following: "cash flow from the new oil rig depends positively on the oil price and market conditions, but the estimated mean cash payoff is so large that the net result is a decrease in the firm's ratio of payoff covariance to payoff mean, thus reducing firm cost of capital". That statement sounds odd but stands on nothing but CAPM logic, under which the ratio of payoff covariance to payoff mean drives the cost of capital and is the minimal sufficient measure of its risk.

Since the mean payoff affects the CAPM cost of capital, and can be a stronger force than the payoff covariance, similar statements should arise throughout business. Typically, for example, it is held that firms with higher operating leverage have higher variability in their results and are therefore more risky and warrant a higher cost of capital. The correct way of thinking in accord with CAPM is to consider what an increase in operating leverage does to the firm's Fama ratio. If the gain in efficiency that comes with the higher fixed costs adds sufficiently to the firm's mean payoff, its Fama ratio and cost of capital might well decrease despite its higher variability or risk. That statement is merely an application of CAPM logic, interpreted correctly.

[17] An important early paper that puts a payoffs interpretation and analysis of CAPM is Stapleton and Subrahmanyam (1978).

# Appendix

This is a numerical example showing how diversifiable risk can be (heavily) priced under CAPM.

Imagine an industry of just two firms $j = A, B$ with industry expected payoff $E[V_{Ind}] = E[V_A] + E[V_B]$ and payoff covariance with the market $\text{cov}(V_{Ind}, V_M)$. Let the industry payoff $V_{Ind}$ occur in one part of amount $50\% V_{Ind} + K$ and the remainder of $50\% V_{Ind} - K$, where $K$ is a known constant. Now suppose that an industry random variable $S \in \{a, b\}$, uncorrelated with the market, decides which firm gets the bigger payoff. Firm $A$ earns the bigger part in the event of $S = a$, and firm $B$ in the event of $S = b$. The *ex ante* probabilities of these events are $p_a = p(S = a) = 1 - p_b$.

The sum of the two firm payoffs is not affected by $S$, and hence the market portfolio "diversifies away" the effect of random influence $S$. In fact, the industry portfolio is sufficient to diversify away all the risk presented by random state $S$.

If $p_a = p_b$, the two firms are identical and must have the same CAPM price, $P_A = P_B$. In that case, $P_A = P_B = \frac{1}{2} P_{Ind}$, and by (1)

$$P_{Ind} = \frac{E[V_{Ind}] - c \ \text{cov}(V_{Ind}, V_M)}{(1 + r_f)}.$$

The industry discount rate under CAPM is therefore $E[r_{Ind}] = E[V_{Ind}]/P_{Ind} - 1$. Adding some numbers for illustration, let $E[V_{Ind}] = 100$, $\text{cov}(V_{Ind}, V_M) = 10^6$, $c = 0.000025, K = 25$ and $r_f = 0$, thus giving $P_{Ind} = 75$ and hence an industry discount rate of $100/75 - 1 = 33\%$. The firm asset prices when $p_a = p_b = 0.5$ are both $75/2 = 37.5$.

The next problem is to find the two asset prices where $p_A \neq p_B$, when the firms are no longer identical but differ only in their diversifiable risk. A standard approach is to find $P_A$ and $P_B$ by discounting the respective expected payoffs at their industry "risk adjusted" discount rate. No allowance is made for their different diversifiable risks, because under CAPM doctrine "diversifiable risk is not a priced risk".

Since random variable $S$ adds only diversifiable risk, we can assume any arbitrary value for $p_a = 1 - p_b$. So assume say $p_a = 0.8$. This probability assessment, based in reality on whatever relevant information is available to the market, has a "numerator effect" but not, it is argued, a "denominator effect".

Continuing for the moment with the conventional approach, we find

$$\begin{aligned} P_A &= \frac{E[V_A]}{(1+r_{Ind})} = \frac{0.8(0.5E[V_{Ind}]+K)+0.2(0.5E[V_{Ind}]-K)}{1.3333} \\ &= \frac{0.6K+0.5E[V_{Ind}]}{1.3333} = \frac{65}{1.3333} = 48.75, \end{aligned}$$

and similarly $P_B = 35/1.3333 = 26.25$. So the two asset prices add up to $P_M = 75$, as above, and all is superficially holding together.

The hidden mistake in these prices is revealed by pricing each asset under the CAPM pricing formula (1). To implement (1) we need the individual asset covariances, $\text{cov}(V_A, V_M)$ and $\text{cov}(V_B, V_M)$, which are not considered in the calculations above. We have enough information already to find these. From the law of complete covariance

$$\begin{aligned} \text{cov}(V_A, V_M) &= E[\text{cov}(V_A, V_M|S)] + \text{cov}(E[V_A|S], E[V_M|S]) \\ &= E[\text{cov}(V_A, V_M|S)], \end{aligned}$$

noting that $\text{cov}(E[V_A|S], E[V_M|S]) = 0$ since the expected market payoff $E[V_M|S] = E[V_M]$ is a constant with respect to $S$.

Hence the two covariances are

$$\begin{aligned} \text{cov}(V_A, V_M) &= E[\text{cov}(V_A, V_M|S)] \\ &= 0.8\ \text{cov}(V_A, V_M|S=a) + 0.2\ \text{cov}(V_A, V_M|S=b) \\ &= 0.8\ \text{cov}(0.5(V_{Ind})+K, V_M) + 0.2\ \text{cov}(0.5(V_{Ind})-K, V_M) \\ &= 0.5\ \text{cov}(V_{Ind}, V_M) = 500000, \end{aligned}$$

and similarly

$$\begin{aligned} \text{cov}(V_B, V_M) &= E[\text{cov}(V_B, V_M|S)] \\ &= 0.2\ \text{cov}(0.5(V_{Ind})+K, V_M) + 0.8\ \text{cov}(0.5(V_{Ind})-K, V_M) \\ &= 0.5\ \text{cov}(V_{Ind}, V_M) = 500000. \end{aligned}$$

So the correct mean-variance CAPM asset prices are

$$P_A = \frac{E[V_B] - c\ \mathrm{cov}(V_A, V_M)}{(1 + r_f)} = 65 - 0.000025(500000) = 52.5, \quad (r_f = 0)$$

and

$$P_B = \frac{E[V_B] - c\ \mathrm{cov}(V_B, V_M)}{(1 + r_f)} = 35 - 0.000025(500000) = 22.5.$$

The sum of the two prices remains the same at $P_M = 75$ $(= 52.5 + 22.5)$, however the individual prices $P_A$ and $P_B$ have now changed. Had we followed one standard line of argument and used the industry discount rate, the resulting asset prices would have been widely different.

This illustration is a clear counter-example to the tenet that diversifiable aspects of firm performance are not priced under CAPM. To the contrary, allowance for the pure chance effect of random variable $S$ leaves asset $A$ with a price-implied discount rate of $E[r_A] = 65/52.5 - 1 = 23.8\%$ and asset $B$ with discount rate of $E[r_B] = 35/22.5 - 1 = 55.6\%$, despite both firms being "pure plays" in the same industry.

The embedded CAPM rationale for why asset $B$ has the higher discount rate is that it has the higher Fama ratio. For asset $A$, this ratio is $\mathrm{cov}(V_A, V_M)/E[V_A] = 500000/65 = 7692.308$ and for asset $B$ it is $\mathrm{cov}(V_B, V_M)/E[V_B] = 500000/35 = 14285.71$. The example is particularly revealing because the two firms have the very same payoff covariance, so the difference in their costs of capital is due entirely to the difference in their payoff means.

Note that the price-weighted industry discount rate is

$$E[r_{Ind}] = \frac{52.5(1.238) + 22.5(1.5556)}{75} - 1 = 33.33\%,$$

exactly as found initially. So the calculations are coherent under CAPM logic.

The moral of this example is that a risk or source of payoff variation that is as uncorrelated with the market as the spin of a wheel, and which is totally eliminated by market diversification, has caused the two firms affected to have very different CAPM discount rates. The implied discount rates $E[r_A]$ and $E[r_B]$ change with any information affecting the market's probability assessment $p_a$, despite $p_a$ having no effect whatsoever on the market risk premium.

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