# Non-Linear Equity Valuation

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We incorporate a real option component into the Ohlson (1995) equity valuation model and then use this augmented model to make assessments about the form and nature of the systematic biases that are likely to arise when empirical work is based on inappropriate linear models of the relationship between the market value of equity and its determining variables. Our empirical analysis, which is based on a broad cross-section of U. K. firms covering the period from 2001 until 2004, confirms that non-linearities do exist in the relationship between the market value of equity and its determining variables at both an aggregate and industry level. We also demonstrate how one can expand equity valuation models in terms of an infinite series of "orthogonal" polynomials and thereby determine the relative contribution which the linear and non-linear components of the relationship between equity value and its determining variables make to overall equity value. This procedure shows that non-linearities in equity valuation can be large and significant, particularly for firms with low earnings to book ratios or where the un-deflated book value of equity is comparatively small.

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## 1. Introduction

Theoretical developments in investment analysis have established that real options have the potential to play an important role in the valuation of capital assets. If a firm has the option of abandoning a poorly performing capital project it can increase the capital project's value well beyond the traditional benchmark given by the present value of its expected future cash flows; likewise, the growth option associated with an unexploited capital project can also be significant when compared with the expected present value of its future cash flows. Moreover, it is now generally accepted that these option values will mean that evaluating capital projects exclusively in terms of the present value of their future cash flows can lead to seriously flawed investment decisions - highly profitable capital projects can be overlooked and poorly performing capital projects wrongly implemented. Given this, it is somewhat surprising that both empirical and analytical work on the relationship between equity value and its determining variables continues to be based on models that determine the value of a firm's equity exclusively in terms of the present value of its future operating cash flows and which, therefore, neglect the real option effects associated with the firm's ability to modify or even abandon its existing operating activities. The Ohlson (1995) model, for example, from which much of the empirical work in the area is motivated (Barth and Clinch, 2005, p. 1), implies that there is a purely linear relationship between the market value of equity and its determining variables. As such it is based on the implicit assumption that real options are of little consequence in the equity valuation process. Given this, it is all but inevitable that when real options do impact on equity values the Ohlson (1995) model will return a systematically biased picture of the relationship between the market value of equity and its determining variables. Fortunately, Ashton, Cooke and Tippett (2003) have generalised the Ohlson (1995) model so that it takes account of the real options that are generally available to firms. Our initial task is to use this more general model to determine the likely form and magnitude of the biases that arise under linear equity valuation models like the one formulated by Ohlson (1995). Having done this, we then employ a broad sample of U. K. firms covering the period from 2001 until 2004 to assess whether the systematic biases predicted by our analysis are compatible with the empirically observed relationship between the market value of equity and its determining variables. Our empirical analysis largely confirms that systematic biases in the form of non-linearities related to scale do arise under the Ohlson (1995) model and moreover, that these non-linearities are consistent with the hypothesis that real options have a significant role to play in the equity pricing process.

Our analysis makes several contributions to the literature. First, we provide empirical evidence on the form and nature of the non-linear relationship that exists between the market value of equity and its determining variables. These non-linearities hold consistently across all industries and so it is unlikely our empirical results can be attributed to just a small group of "rogue" industrial classifications. Second, we introduce an hitherto unused "orthogonal polynomial" expansion procedure for identifying the relative contribution which the linear and non-linear components of the relationship between equity value and its determining variables make to overall equity value. Suffice it to say the evidence from this procedure is that non-linearities in equity valuation can be large and significant, particularly for firms with low earnings to book ratios or where the un-deflated book value of equity is comparatively small. Third, we demonstrate that there ought to be an inverse relationship between the market to book ratio and the un-deflated book value of equity - reflecting the fact that larger firms will be able to "ride out" adverse economic circumstances more effectively than smaller firms. Our empirical analysis demonstrates that our data is compatible with this hypothesis – something that has not previously appeared (or been demonstrated) in the literature.

In the next section we briefly summarise the main features of the Ashton, Cooke and Tippett (2003) equity valuation model and in particular, some important scale invariance principles on which it is based. Recall here that empirical work in the area is invariably based on market and/or accounting (book) variables that have been "normalised" or "deflated" in order to facilitate comparisons between firms of different size. Given this, it is important that one appreciates how these deflation procedures might alter or even distort the underlying "levels" relationship which exists between the market value of equity and its determining variables. In section 3 we summarise our empirical evidence on the relationship between the market value of equity and its determining variables and show how this confirms that the non-linearities which appear to affect pricing relationships on the U.K. equities market are compatible with the Ashton, Cooke and Tippett (2003) equity valuation model. In section 4 we demonstrate how equity valuation models can be expanded in terms of an infinite series of "orthogonal" polynomials and how this in turn allows one to assess the relative contribution which the linear and non-linear components of the relationship between equity value and its determining variables make to overall equity value. Section 5 contains our summary conclusions.

## 2. Real Options and Equity Value

The Ashton, Cooke and Tippett (2003) model is based on the assumption that there are two components to the market value of a firm's equity. The first is called the recursion value of equity and is the present value of the future cash flows the firm expects to earn given that its existing investment opportunity set will be maintained indefinitely into the future. The Ohlson (1995) equity valuation model is exclusively based on this component of equity value. There is, however, a second component of equity value which the Ohlson (1995) model neglects; namely, the real option (or adaptation) value of equity. This is the option value that arises from a firm's ability to change its existing investment opportunity set by (for example) fundamentally changing the nature of its operating activities (Burgstahler and Dichev, 1997, p. 188). Ashton, Cooke and Tippett (2003) employ this distinction to develop a quasi-supply side generalisation of the Ohlson (1995) model under which the recursion value of equity, $\eta(t)$, evolves in accordance with the stochastic differential equation:

$$d\eta(t) = i\eta(t)dt + \eta^{\delta}(t)dq(t) \qquad (1)$$

where i is the cost of equity capital, dq(t) is a Wiener process with a variance parameter of $\zeta$ and $0 \leq \delta \leq \frac{1}{2}$ is a real number. It then follows that in expectations the proportionate rate of growth in the recursion value of equity will be equal to the cost of equity, or $E_t[d\eta(t)] = i\eta(t)dt$ where $E_t(\cdot)$ is the expectations operator taken at time t. Furthermore, the variance of instantaneous increments in the recursion value of equity will be $Var_t[d\eta(t)] = \eta^{2\delta}Var_t[dq(t)] = \eta^{2\delta}(t)\zeta dt$. Finally, note that when $\delta = 0$, the recursion value of equity evolves in terms of an Uhlenbeck and Ornstein (1930) process whilst when $\delta = \frac{1}{2}$ recursion value evolves in terms of a continuous time branching process (Feller, 1951, pp. 235-237; Cox and Ross, 1976, p. 149). Moreover, since the literature normally describes the firm's investment opportunity set in terms of a first order autoregressive system of "information" variables it also follows that the recursion value of equity, $\eta(t)$, will be a linear sum of more "primitive" variables. A good example of this is provided by the Ohlson (1995, pp. 667-669) linear information dynamics which implies that the recursion value of equity evolves in terms of the book value of equity, $b(t)$, "book" or accounting earnings, $x(t)$, and an "information" variable, $\nu(t)$, in accordance with the following formula:

$$\eta(t) = c_1 b(t) + c_2 x(t) + c_3 \nu(t) \tag{2}$$

where $c_1$, $c_2$ and $c_3$ are the relevant valuation coefficients.

Ashton, Cooke and Tippett (2003) employ the quasi-supply side model summarised above in conjunction with standard no arbitrage conditions and thereby show that the market value of the firm's equity, $P(\eta,B,\theta,\delta)$, will have to be:

$$P(\eta,B,\theta,\delta) = \eta + \frac{B}{2}\int_{-1}^{1} \exp\left\{\frac{-\theta\left(\frac{2\eta}{1+z}\right)^{2(1-\delta)}}{2(1-\delta)}\right\}dz \tag{3}$$

Here $B > 0$ denotes the value of the firm's adaptation options when the recursion value of equity falls away to nothing and $\theta = \frac{2i}{\mathrm{Var}_t[dq(t)]} = \frac{2i}{\zeta}$ is a "risk parameter" that captures the relative stability with which the recursion value of equity evolves over time. Note that $\eta$, the first term on the right hand side of this equation, is the recursion value of equity on which the Ohlson (1995) equity valuation model is exclusively based. We have previously noted, however, that firms invariably have the option of changing their investment opportunity sets and this gives rise to a real option component to equity value that is captured by the integral term in the above valuation formula. Here it is important to note that as the variability ($\zeta$) of the recursion value increases relative to the cost of equity (i), the term $\exp\left\{\frac{-\theta\left(\frac{2\eta}{1+z}\right)^{2(1-\delta)}}{2(1-\delta)}\right\}$ grows in magnitude and the real option value of equity becomes larger as a consequence. Similarly, the real option value of equity falls as the variability of the recursion value declines relative to the cost of equity. In other words, when the rate of growth in recursion value clusters closely around the cost of equity it is unlikely the catastrophic events which will induce the firm to exercise its real options will arise. In these circumstances, the small probability of these options ever being exercised will mean that the real option value of equity will also have to be comparatively small.

Here we need to note, however, that empirical work in the area is invariably based on market and/or accounting (book) variables that have been "normalised" or "deflated" in order to facilitate comparisons between firms of different size. Given this, suppose one defines the normalised recursion value $h(t) = \frac{\eta(t)}{w}$ where w is some kind of normalising factor. It then follows that increments in the normalised recursion value will evolve in terms of the process:

$$dh(t) = ih(t)dt + h^{\delta}(t)dv(t) \tag{4}$$

where $dv(t) = w^{\delta - 1}dq(t)$ is a Wiener process with variance parameter $w^{2(\delta - 1)}\zeta$. Note that increments in the normalised recursion value, h(t), will have a mean and variance of $E_t[dh(t)] = ih(t)dt$ and $Var_t[dh(t)] = h^{2\delta}(t)Var_t[dv(t)] = h^{2\delta}(t)w^{2(\delta - 1)}\zeta dt$, respectively. This in turn implies that the risk parameter for the normalised recursion value will be $\frac{2i}{Var_t[dv(t)]} = \frac{2i}{w^{2(\delta - 1)}\zeta} = \theta w^{2(1 - \delta)}$. Furthermore, if one works in terms of this normalised variable, h(t), rather than in levels, η(t), it follows that the value of equity will have to be:

$$P(h, \frac{B}{w}, \theta w^{2(1-\delta)}, \delta) = h + \frac{B}{2w}\int_{-1}^{1} \exp\left\{\frac{-\theta w^{2(1-\delta)}\left(\frac{2h}{1+z}\right)^{2(1-\delta)}}{2(1-\delta)}\right\} dz \tag{5}$$

Now here it is important to note that η(t) = wh(t) and so, the normalised value of equity will satisfy the following important property:

$$wP(h,\frac{B}{w},\theta w^{2(1-\delta)},\delta) = \eta + \frac{B}{2}\int_{-1}^{1} \exp\{\frac{-\theta(\frac{2\eta}{1+z})^{2(1-\delta)}}{2(1-\delta)}\}dz = P(\eta,B,\theta,\delta) \quad (6)$$

Formally, this result means that the Ashton, Cooke and Tippett (2003) equity valuation model is scale-invariant under all dilations, w [Borgnat, Flandrin and Amblard (2002, p. 181)]. [1]

Now consider a firm for which all variables have been deflated by the book value of equity, w = B > 0, as at some *fixed* date or that $h(t) = \frac{\eta(t)}{B}$ in the scaled version of the Ashton, Cooke and Tippett (2003) model given earlier. Moreover, assume, for purposes of illustration, that the firm's adaptation options involve selling off its existing investment opportunities at their book values as recorded on the balance sheet and using the proceeds to move into alternative lines of business. [2] Finally, suppose the recursion

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1 If, for all real x, λ and some Δ, a function, f(·), satisfies the property $\lambda^{\Delta}f(x) = f(\lambda x)$ then it is said to be scale invariant for all dilations, λ, with a scaling dimension of Δ [Borgnat, Flandrin and Amblard (2002, p. 181)]. See Alexander and Nogueira (2006, pp. 4-6) for examples of option models that violate the scale-invariance property given here. Note also that the scale invariance property demonstrated here can be used to develop other properties of the equity pricing relationship. As an example of this, suppose one defines $\varepsilon_{P;\eta} = \frac{\eta}{P}\cdot\frac{\partial P}{\partial \eta}$ as the equity security's price elasticity with respect to instantaneous changes in its recursion value, $\varepsilon_{P;B} = \frac{B}{P}\cdot\frac{\partial P}{\partial B}$ is defined as the equity security's price elasticity with respect to instantaneous changes in its liquidation value and $\varepsilon_{P;\theta} = \frac{\theta}{P}\cdot\frac{\partial P}{\partial \theta}$ is defined as the equity security's price elasticity with respect to instantaneous changes in the risk parameter. Using the given scale invariance property one can then show:

$$\varepsilon_{P;\eta} + \varepsilon_{P;B} = 1 + 2(1 - \delta)\varepsilon_{P;\theta}$$

or that there is a linear relationship between the three price elasticity measures. Moreover, the elasticity measures given here are dimensionless statistics that are directly comparable across firms without further adjustment.

2 This assumption is widely used in the empirical work of the area. Burgstahler and Dichev (1997, p. 195), for example, define the book value of equity at the *beginning* of the interval covered by the profit and loss statement "as the measure of adaptation value for the year." Barth, Beaver and Landsman (1998, pp. 1-2), Collins, Pincus and Xie (1999, p. 32), Ashton, Cooke and Tippett (2003, p. 427) and Cotter and Donnelly (2006, p. 11) also invoke this assumption. One could, however, define $h(t) = \frac{\eta(t)}{b(t)}$ so that the recursion value of equity is scaled by contemporaneous (and not lagged) book value (that is, the denominator and the numerator are evaluated at the same point in time). It is not hard to show that under this specification the scaled recursion value of equity evolves in terms of the process:

value of equity evolves in terms of a continuous time branching process in which case $\delta = \frac{1}{2}$. These considerations will mean the market value of equity (per unit of book value) evolves in terms of the following formula: [3]

$$\frac{P(\eta,B,\theta,\frac{1}{2})}{B} = h + \frac{1}{2}\int_{-1}^{1} \exp(\frac{-2\theta Bh}{1 + z})dz = P(h,1,\theta B,\frac{1}{2}) \qquad (7)$$

Suppose now that one mistakenly assumes the existence of a linear relationship between the market value of equity and its determining variables, as is the case with the Ohlson (1995) model. These linear models are based on the implicit assumption that the real options generally available to firms have no role to play in the equity valuation process. However, if one uses a linear model to approximate the relationship between equity value and its determining variables when in fact, real options do impact on equity values then it is all but inevitable there will be systematic differences between the actual market value and those "predicted" by the linear model. One can illustrate the point being made here by approximating the market to book ratio given in the scaled version of the Ashton, Cooke and Tippett (2003) model given above by a linear function over its entire domain. Our approximating procedures make use of an inner product (Hilbert) space using the Laguerre polynomials as a basis and are summarised in further detail in the Appendix. These procedures show that the "best" linear approximation to the equity valuation function $[P(h,1,\theta B,\frac{1}{2})]$ over the semi-infinite real line will be:

$$P(h,1,\theta B,\frac{1}{2}) \approx \frac{1}{1 + \theta B} + [1 + \frac{\theta B}{1 + \theta B} + \theta B\log(\frac{\theta B}{1 + \theta B})].h \qquad (8)$$

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$$dh(t) = (i - r(t))h(t)dt + h^{\delta}(t)dv(t)$$

where $r(t) = \frac{x(t)}{b(t)}$ is the instantaneous book or accounting rate of return and $dv(t) = [b(t)]^{\delta - 1}dq(t)$ is a Wiener process with variance parameter $[b(t)]^{2(\delta - 1)}\zeta$. Unfortunately, one can neither solve this differential equation nor determine the distributional properties of the scaled recursion value, h(t), implied by it.

[3] The integral term in this expression, which captures the real option value of equity, cannot be evaluated in terms of elementary functions. However, it is still possible to evaluate the integral term numerically. In the appendix we provide a table of values for this integral and demonstrate how it is used.

As a particular example consider a firm whose normalised risk parameter is $\theta B = 2$ in which case substitution shows that the best linear approximation to the equity valuation function will be: [4]

$$P(h,1,2,\tfrac{1}{2}) = h + \frac{1}{2}\int_{-1}^{1} \exp\left(\frac{-4h}{1+z}\right)dz \approx 0.3333 + 0.8557h \qquad (9)$$

Figure 1 contains a diagrammatic summary of these results. The upward sloping line

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FIGURE ONE ABOUT HERE

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emanating from the origin at a 45 degree angle is the normalised recursion value of equity, $h = \frac{\eta}{B}$. The downward sloping curve which asymptotes towards the recursion value axis is the normalised real option value of equity, $\frac{1}{2}\int_{-1}^{1} \exp\left(\frac{-4h}{1+z}\right)dz$. The sum of the normalised recursion and normalised real option values is the market value of equity per unit of book value, $P(h,1,2,\tfrac{1}{2}) = h + \frac{1}{2}\int_{-1}^{1} \exp\left(\frac{-4h}{1+z}\right)dz$, and is represented by the convex curve which asymptotes towards the 45 degree line representing the normalised recursion value of equity. Now here it is important to observe that as the normalised recursion value increases in magnitude, the market value of equity (per unit of book value) at first declines before reaching a minimum and then gradually increasing in magnitude. This arises because at small recursion values the decline in real option value will be much larger than the increase in the recursion value itself. This in turn means that the best linear approximation, $0.3333 + 0.8557h$, to the overall market value of equity will bear a

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[4] Ataullah, Higson and Tippett (2006) summarise empirical evidence which is broadly compatible with this value of the normalised risk parameter, $\theta B$.

particular relationship to the market value of equity. Hence, when the normalised recursion value of equity is "low" ($h < 0.30$) then the difference between the linear approximation and the market value of equity, $(0.3333 + 0.8557h) - P(h,1,2,\frac{1}{2})$, is negative. Beyond this point (that is, $h > 0.30$) the difference between the linear approximation and the market value of equity becomes positive before falling away again and becoming negative when $h > 2.31$ (although this is not shown on the graph). In other words, for individual firms there will be systematic biases in the linear model which is used to approximate the relationship between the market value of equity and its determining variables. [5]

Here we need to emphasise, however, that the form and nature of the systematic biases which arise from approximating the relationship between equity value and its determining variables in terms of a linear model will very much depend on the magnitude of the normalised risk parameter, $\theta B$. The smaller this parameter (and by implication the larger the real option values) the more likely it is that a linear model will provide a "good" approximation to the relationship between the market value of equity and its determining variables. This is illustrated by Figure 2 which shows for a firm whose

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FIGURE TWO ABOUT HERE

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normalised risk parameter is $\theta B = 0.25$, that the best linear approximation to the equity valuation function will be:

$$P(h,1,0.25,\tfrac{1}{2}) = h + \frac{1}{2}\int_{-1}^{1} \exp\left(\frac{-0.5h}{1+z}\right)dz \approx 0.8000 + 0.7976h \qquad (10)$$

[5] It is somewhat surprising that linear valuation models continue to dominate analytical and empirical work in this area given that Black and Scholes (1973, pp. 649-652) and Cox and Ross (1976, pp. 163-165) demonstrated over thirty years ago that it is highly unlikely such models can provide an adequate description of the way equity prices evolve in practice.

Note that with a relatively small normalised risk parameter like this it is only at very small and very large ratios of the recursion to book value of equity that linear approximations will provide a poor reflection of the relationship between the market value of equity and its determining variables.

One can further illustrate the importance of the systematic biases demonstrated in these examples by thinking of the valuation equation $P(h,1,2,\frac{1}{2})$ as a "representative firm" in a large cross sectional sample of similarly prepared firms. By "similarly prepared" is meant that all firms are characterised by a common investment opportunity set and are, therefore, described by a common equity valuation function; namely, $P(h,1,2,\frac{1}{2})$. [6] It then follows that if equity values include a real option component cross sectional linear regression models of the relationship between equity prices and recursion values will follow a pattern similar to that obtained for the above examples. That is, one would expect to find firms with low ratios of recursion value to the book value of equity returning negative residuals from a linear regression model. Likewise, firms with intermediate ratios of recursion value to the book value of equity will return positive residuals from the linear regression model. Finally, when the ratio of recursion value to the book value of equity is large one would expect to see negative residuals again emerging from the linear regression model. Unfortunately, these implications of the Ashton, Cooke and Tippett (2003) model are of limited empirical significance because of the fact that the recursion value of equity, $\eta$, is an unobservable variable.

One can demonstrate this latter point by recalling from equation (2) that under the Ohlson (1995, pp. 667-669) model the recursion value of equity evolves in terms of the formula:

$$\eta(t) = c_1 b(t) + c_2 x(t) + c_3 \nu(t)$$

where $b(t)$ is the book value of equity, $x(t)$ is the firm's earnings, $\nu(t)$ is an information variable and $c_1$, $c_2$ and $c_3$ are the relevant valuation coefficients. However, whilst

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[6] This approach underscores much of the empirical work conducted in the area. Dechow, Hutton and Sloan (1999) and Collins, Pincus and Xie (1999) and Morel (2003) provide some recent examples.

earnings and the book value of equity are easily observed from a firm's financial statements, Myers (1999, p. 8) (amongst others) notes that "it is not possible to explicitly control for all possible [interpretations of] $\nu(t)$". Fortunately, there are ways of addressing the difficulties this poses for empirical work. One can illustrate this by taking the total derivative through the Ashton, Cooke and Tippett (2003) equity valuation model in which case it follows $dP = \frac{\partial P}{\partial \eta} d\eta$ is the instantaneous change in the market value of equity induced by an instantaneously small change, $d\eta$, in the recursion value of equity (Spiegel, 1974, p. 105). However, under the Ohlson (1995) model instantaneous changes in the recursion value of equity are related to instantaneous changes in its determining variables through the formula:

$$d\eta(t) = c_1 x(t)dt + c_2 dx(t) + c_3 d\nu(t)$$

where $dx(t)$ is the instantaneous change in earnings, $d\nu(t)$ is the instantaneous change in the information variable and the clean surplus identity will mean that $db(t) = x(t)dt$ is the instantaneous change in the book value of equity. It then follows that instantaneous changes in the market value of equity are related to instantaneous changes in its determining variables through the formula:

$$dP = \frac{\partial P}{\partial \eta} d\eta = (c_1 x dt + c_2 dx + c_3 d\nu)\frac{\partial P}{\partial \eta} \qquad (11)$$

Now, suppose one holds all variables constant except for earnings in which case one has $dt = 0 = d\nu$ and $dP = c_2 . \frac{\partial P}{\partial \eta} dx$ or $\frac{dP}{dx} = c_2 . \frac{\partial P}{\partial \eta}$. This in turn means that a graph of the market value of equity against earnings will, except for a scaling factor, be equivalent to the graph of the market value of equity against its recursion value as for instance, in Figure 1 above. Hence, if after controlling for these "other" factors which affect equity prices one finds that as earnings grow the market value of equity at first declines but then gradually increases in magnitude then one will have indirect evidence that real options have an important role to play in the determination of equity prices. With this in mind we now review previous empirical work in the area before summarising our own empirical evidence on the real options issues raised here. Our empirical analysis shows that the non-linearities in equity valuation hold consistently across all industries and so it is unlikely that the aggregate results previously reported in the literature can be attributed to just a small group of "rogue" industrial classifications. We also demonstrate empirically that there is an inverse relationship between the market to book ratio and the un-deflated book value of equity - reflecting the fact that larger firms are able to "ride out" adverse economic circumstances more effectively than smaller firms.

## 3. Data and Empirical Evidence

The early evidence of Burgstahler and Dichev (1997, pp. 199-205) and Burgstahler (1998, p. 339) shows that at an aggregate level there is a highly convex relationship between the ratio of the market value of equity to the book value of equity and the ratio of earnings attributable to equity to the book value of equity for U.S. equity securities over the twenty year period ending in 1994. Likewise, Ashton, Cooke and Tippett (2003, pp. 429-430) show that an almost identical convex relationship exists at an aggregate level between the market value of equity and the earnings attributable to equity for U.K. equity securities over the period from 1987 to 1998. Finally, Di-Gregorio (2006) shows that there is a highly convex relationship (again at an aggregate level) between the market value of equity and earnings for German and Italian firms over the period from 1995 to 2005. In other words, there is now a large volume of international empirical evidence in support of the hypothesis that at an aggregate level there is a convex relationship between the market value of equity and earnings.

Further evidence on the relationship between the market value of equity and earnings for U.K. firms was obtained by extracting earnings, the market value of equity and the book value of equity for all U.K. firms summarised on the Datastream system over the period from 2001 until 2004. This was a period of depressed equity prices in the U.K. in contrast to the mainly buoyant conditions on which much of the previously summarised empirical work is based. The market value of equity for a particular firm is defined as its share price multiplied by the number of shares on issue 100 days after its balance sheet date.[7] Earnings is the firm's net income before exceptional items less any

[7] Our intention here is that equity prices should reflect the most recent earnings figures available. However, Cotter and Donnelly (2006, p. 27) note that "U.K. companies report more tardily than their U.S. counterparts …." Given this, we also replicated the analysis summarised in this section using equity prices 180 days (6 months) after the given balance sheet dates. However, since the results are virtually identical to those reported in the text we do not discuss them further here.

preference dividends over the year covered by its balance sheet date. Finally, the book value of equity is defined as the book value of equity at the beginning of the year covered by the firm's balance sheet date less the book value of intangible assets also at the beginning of the year covered by the firm's balance sheet date. [8] Here Figure 3 provides a graphical summary of the relationship between the market value of equity and earnings (both normalised by the book value of equity) for all publicly listed U.K. firms on the Datastream system over the period from 2001 until 2004 and is based on 4,043 firm-year observations. The broadly convex nature of the relationship depicted here is similar to

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FIGURE THREE ABOUT HERE

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that obtained in previous empirical work in the area and is consistent with the hypothesis that real options have a significant role to play in determining equity prices.

The convex nature of the relationship between the market value of equity and earnings is further emphasised by Figure 4. The data on which Figure 4 is based were

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FIGURE FOUR ABOUT HERE

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obtained by ordering the 4,043 earnings to book ratios from the smallest to the largest ratio. The simple average of the lowest 75 earnings to book ratios was then calculated (-1.0687) together with the simple average of their market to book ratios (4.9333). The average of the next 75 lowest earnings to book ratios was then calculated (-0.8393) together with the average of their market to book ratios (4.1193). This procedure was

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[8] We also replicated the empirical analysis reported here with book values that included intangible assets. Consistent with other empirical work in the area (Ashton, Cooke and Tippett, 2003, p. 436), this did not result in any qualitative differences when compared with the results reported in the text.

continued until the entire file was "exhausted" and is designed to reduce the impact which variables other than earnings have on the market value of the equity securities. [9]

Note here, however, that the averaged ordered pairs plotted in Figure 4 reveal that the convex relationship between the market value of equity and earnings is both very strong and consistent with the hypothesis that real options play a significant role in the determination of equity prices. Here it will be recalled that at large negative earnings to book ratios our prediction is that the decline in real option value will be much greater than the increase in the recursion value of equity. This in turn means that as the earnings to book ratio grows the market to book ratio will at first decline before reaching a minimum and then gradually increasing in magnitude. Moreover, this also means it is all but inevitable there will be systematic biases in empirical work that assumes the existence of a purely linear relationship between the market value of equity and its determining variables (Dechow, Hutton and Sloan, 1999; Collins, Pincus and Xie, 1999; Morel, 2003). Hence, from Figure 4 it is hardly surprising that the residuals from a linear regression of the market value of equity against earnings are negative for low values of the earnings to book ratio, positive for intermediate values of the earnings to book ratio and then negative again for large values of the earnings to book ratio. [10]

It also needs to be emphasised that there is very little difference between the aggregate results summarised in Figures 3 and 4 and the results for the eighteen industry classifications on which our aggregate results are based. In other words, it is unlikely the convex relationship between the market value of equity and earnings exhibited by Figures 3 and 4 can be attributed to just a small number of "rogue" industrial classifications. As evidence of this, Figures 5 through 10 summarise the relationship between earnings and the market value of equity for three industrial classifications that are typical of our results across all 18 industrial classifications on which our aggregate results are based. Here Figure 5 summarises the raw data for the U.K. Financial Services Industry and is based

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[9] See the Appendix where we show that the limiting value of the information variable, $\nu(t)$, for the Ohlson (1995) equity valuation model under the averaging procedure employed here will be zero, almost surely.

[10] Akbar and Stark (2003) also document systematic biases similar to those illustrated here in a more comprehensive linear model of the relationship between the market value of equity and its determining variables, albeit in a slightly different context to the one considered here.

on 571 firm-year observations covering the period from 2001 until 2004. Note that the convex relationship between earnings and the market value of equity exhibited in Figure

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FIGURE FIVE ABOUT HERE

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5 is very similar to the convex relationship between these variables for the aggregate data summarised in Figure 3. Likewise, the convex relationship between earnings and the market value of equity for the "grouped" data exhibited in Figure 6 is very similar to the relationship for the grouped aggregate data summarised in Table 4.[11] As with the aggregate grouped data summarised in Figure 4, the average ordered pairs plotted in Figure 6 reveal that the convex relationship between the market value of equity and earnings for the Financial Services Industry is both very strong and consistent with the hypothesis that real options have a significant impact on equity prices.

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FIGURE SIX ABOUT HERE

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Figures 7 and 8 summarise the raw and grouped data relationships respectively between the market to book ratio and the earnings to book ratio for equity for the Industrial Goods and Services industry classification. Figures 9 and 10 provide the same information for the Technology industrial classification. Note again that the convex relationship between earnings and the market value of equity exhibited in Figure 7 and 9 is similar to the convex relationship between these variables for the aggregate data summarised in Figure 3. Likewise, the convex relationship between earnings and the market value of equity for the grouped data summarised in Figure 8 and 10 is very similar to the relationship for the aggregate grouped data summarised in Figure 4. In particular Figures 8 and 10 show that the residuals from a linear regression of the market value of

[11] For Figure 6 the earnings to book ratios were grouped into 26 ordered sub-groups comprising about 22 firm-years each. The average earnings to book ratio and average market to book ratio was then computed for each of the 26 groups. It is these 26 ordered pairs that are plotted as Figure 6. Similar procedures were used in determining the data plotted in Figures 8 and 10.

equity against earnings are negative for low values of the earnings to book ratio, positive for intermediate values of the earnings to book ratio and negative again for large values

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FIGURES SEVEN, EIGHT, NINE AND TEN ABOUT HERE

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of the earnings to book ratio. In other words, for these industries too there is evidence that real options play a significant role in determining equity prices and because of this, there will be systematic biases in models that assume a purely linear relationship between the market value of equity and earnings.

Further evidence on the impact that real options have on equity values is to be had from the scaled version of the Ashton, Cooke and Tippett (2003) model given earlier which from equation (7) it will be recalled takes the form:

$$P(h,1,\theta B,\tfrac{1}{2}) = h + \frac{1}{2}\int_{-1}^{1} \exp\left(\frac{-2\theta Bh}{1+z}\right)dz = \frac{P(\eta,B,\theta,\tfrac{1}{2})}{B}$$

where $P(h,1,\theta B,\tfrac{1}{2})$ is the normalised market value of equity, $\eta$ is the recursion value of equity, B is the book (or liquidation) value of equity, $h = \frac{\eta}{B}$ is the normalised recursion value of equity and $\theta$ is the risk parameter that captures the relative stability with which the recursion value of equity evolves over time. Note that under this formula the smaller of any two otherwise identical firms (in terms of the book value of equity, B) will have the higher normalised equity value. The reason for this is that larger firms will be able to "ride out" adverse economic circumstances more effectively than smaller firms. This in turn means that there is a greater probability that smaller firms will have to exercise their real options and so, these options become more valuable as a consequence [Ashton, Cooke, Tippett and Wang (2004, pp. 291-292)]. Moreover, one can test this prediction that the smaller of two otherwise identical firms will have a larger normalised equity value by employing an aggregation procedure similar to that on which previous analysis is based. Given this, we ordered the 4,043 market to book ratios on which our empirical analysis is based from the smallest to the largest ratio. The simple average of the lowest 75 market to book ratios was then calculated together with the simple average of their un-deflated book values. The average of the next 75 smallest market to book ratios was then calculated together with the average of their un-deflated book values. We continued with this procedure until all 4,043 market to book ratios had been included in an ordered aggregate portfolio. These averaged ordered pairs were then summarised graphically as Figure 11. Note how the graph confirms that there is a downward sloping relationship

FIGURE ELEVENABOUT HERE

between the market to book ratio and the un-deflated book value of equity. Thus, as with previous empirical evidence summarised in this paper our analysis here is compatible with the hypothesis that real options have a significant role to play in the determination of equity prices.

## 4. The Relative Importance of Non-Linearities in Equity Valuation

Our previous analysis establishes the inevitability of systematic biases in models that presume a purely linear relationship between the market value of equity and its determining variables. We now investigate whether it is possible to isolate the contribution which the linear and non-linear components of the relationship between equity value and its determining variables make to overall equity value and in particular, whether it might be possible to characterise equity value in terms of a low order polynomial expansion of its determining variables. We begin by noting that the inner product (Hilbert) space framework employed earlier can be used to express equity value as an infinite series of orthogonal (that is, uncorrelated) polynomial terms of the determining variables and this in turn, allows one to ascertain the relative contribution which each polynomial term makes to the overall variation in equity value. One can illustrate the point being made here by noting that the inner product (Hilbert) space framework implies that the Ashton, Cooke and Tippett (2003) equity valuation model,

$P(h,1,B\theta,\frac{1}{2})$, can be expressed in terms of an infinite series expansion of Laguerre polynomials, namely:[12]

$$P(h,1,B\theta,\tfrac{1}{2}) = \sum_{n=0}^{\infty} \alpha_n L_n(h) \qquad (12)$$

where $L_0(h) = 1$, $L_1(h) = 1 - h$ and when $n \geq 2$,

$nL_n(h) = (2n - 1 - h)L_{n-1}(h) - (n - 1)L_{n-2}(h)$ are the Laguerre polynomials.

Furthermore, $\alpha_0 = \theta B\log(\frac{\theta B}{1 + \theta B}) + 2$, $\alpha_1 = -1 - \frac{B\theta}{1 + B\theta} - \theta B\log(\frac{\theta B}{1 + \theta B})$ and when $n \geq 2$,

$\alpha_n = \frac{B\theta(1 + B\theta)^n - (B\theta)^n(n + B\theta)}{n(n - 1)(1 + B\theta)^n}$ are the coefficients associated with each of the Laguerre polynomials in the series expansion.

Now suppose one approximates the equity valuation function as a linear sum of the first (m + 1) Laguerre polynomials, or:

$$P(h,1,B\theta,\tfrac{1}{2}) \approx \sum_{n=0}^{m} \alpha_n L_n(h) \qquad (13)$$

It can then be shown that $\alpha_0^2.[\sum_{n=0}^{\infty} \alpha_n^2)]^{-1}$ gives the proportion of the squared variation in the equity valuation function, $P(h,1,B\theta,\frac{1}{2})$, which is accounted for by the Laguerre polynomial $L_0(h) = 1$ (Apostol, 1967, p. 566). Likewise, $\alpha_1^2.[\sum_{n=0}^{\infty} \alpha_n^2)]^{-1}$ gives the proportion of the squared variation in the equity valuation function which is accounted for by the Laguerre polynomial $L_1(h) = 1 - h$. Similarly, $\alpha_2^2.[\sum_{n=0}^{\infty} \alpha_n^2)]^{-1}$ gives the proportion of the squared variation which is accounted for by the Laguerre polynomial $L_2(h) = \frac{1}{2}(h^2 - 4h + 2)$. Continuing this procedure shows that:

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[12] The proofs for this and subsequent results appear in the Appendix.

$$R_m^2 = \sum_{n=0}^{m} \alpha_n^2 . [\sum_{n=0}^{\infty} \alpha_n^2)]^{-1} \qquad (14)$$

gives the proportion of the squared variation in the equity valuation function which is accounted for by the first (m + 1) Laguerre polynomials. We now show that one can use these results to determine the relative contribution which the linear and non-linear components of the equity valuation function make to overall equity value. [13]

Table 1 summarises the relative contribution which the Laguerre polynomials of order m = 0 to m = 100 make to the overall squared variation of the Ashton, Cooke and Tippett (2003) equity valuation function, $P(h,1,B\theta,\frac{1}{2})$ for values of the scaled risk parameter that vary from θB = 0.25 to θB = 8. [14] Note how the Table shows that a linear Approximation ($R_1^2$), based on the coefficients $\alpha_0$ and $\alpha_1$, accounts for over 98% of the squared variation of the Ashton, Cooke and Tippett (2003) equity valuation function, irrespective of the scaled risk parameter, θB, on which the approximation is based. If, for example, one follows the analysis in Section 2 by letting θB = 2 then Table 1 shows $\alpha_0 = 1.1891$ and $\alpha_1 = -0.8557$ or that consistent with equation (9), the best linear

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TABLE ONE ABOUT HERE

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approximation to the equity valuation function is:

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13 The procedure articulated here is equivalent to the spectral decomposition of a variance-covariance matrix using the method of Principal Components (Rao, 1964). The squared coefficients, $\alpha_n^2$, are equivalent to the eigenvalues of the variance-covariance matrix. In the same way as the ratio of a given eigenvalue to the sum of all eigenvalues (the spur of the matrix) gives the proportion of the variance accounted for by the eigenvector (principal component) corresponding to the given eigenvalue, the ratio of the squared coefficient to the sum of all squared coefficients gives the proportion of the squared variation in the equity valuation function accounted for by the particular Laguerre polynomial.

14 We have previously noted that Ataullah, Higson and Tippett (2006) summarise empirical evidence which is broadly compatible with these values of the normalised risk parameter, θB.

$$P(h,1,2,\tfrac{1}{2}) = h + \frac{1}{2}\int_{-1}^{1} \exp(\frac{-4h}{1+z})dz \approx \alpha_0 + \alpha_1(1 - h) = 0.3333 + 0.8557h$$

Moreover, Table 1 also shows that this simple linear approximation accounts for $R_1^2 = 98.4652\%$ of the squared variation in $P(h,1,2,\frac{1}{2})$. This has the important implication that the non-linear terms in the polynomial expansion account for no more than $1 - R_1^2 = 1 - 0.984652 = 1.5348\%$ of the squared variation in $P(h,1,2,\frac{1}{2})$. A similar conclusion applies for other values of the scaled risk parameter, θB, summarised in Table 1; the non-linear terms in the polynomial expansion make only a minor contribution to the squared variation in the equity valuation function. Given this, one might conclude that a linear approximation of the relationship between equity value and its determining variables will suffice for most practical purposes. Unfortunately, the "least squares" procedures employed here and also in most of the empirical work of the area, suffer from a "ringing artifact" known as Gibbs' phenomenon. [15] In the present context Gibbs' phenomenon implies that the Laguerre series expansion will display irregular behaviour in an arbitrarily small interval near the origin (Fay and Kloppers, 2006). This in turn means that whilst a linear approximation will provide generally reasonable estimates of the equity valuation function when the recursion value of equity is comparatively "large", it will, unfortunately, perform poorly near the origin (that is, when the recursion value of equity is relatively small). It is in this latter circumstance that one will need to include higher order terms from the series expansion if there is to be any prospect of obtaining reasonable approximations to the equity valuation function - something that is borne out by the empirical work summarised in the previous section of this paper.

The exact degree to which the Laguerre polynomial series expansion must be carried before one obtains reasonable approximations to the equity valuation function near the origin very much depends on the magnitude of the scaled risk parameter, Bθ. Larger values of this parameter will generally require the inclusion of higher order polynomial terms. Here one can again follow the analysis in Section 2 in letting Bθ = 2

[15] The "least squares" techniques on which much of the empirical work of the area is based can also be formalised in terms of an inner product (Hilbert) space with a Euclidean norm and will, as a consequence, also be affected by Gibbs' phenomenon.

in which case Figure 12 plots the equity valuation function, $P(h,1,2,\frac{1}{2})$, together with its fifth (m = 5), tenth (m = 10), fifteenth (m = 15) and twentieth (m = 20) degree Laguerre polynomial series approximations. Note how the fifth degree Laguerre approximation is particularly poor near the origin and that the tenth and fifteenth degree approximations, whilst an improvement, are still not entirely satisfactory. Indeed, it is only when one

FIGURE TWELVE ABOUT HERE

employs a twentieth degree polynomial expansion that the approximation to the equity valuation function becomes at all reasonable near the origin. The important point here is that even though linear approximations may appear to be more than reasonable over virtually the entire domain of equity values, nonetheless near the origin (where the recursion value of equity is small) they can be especially poor. This means that mis-specification errors are more likely with samples comprised of firms with comparatively small recursion values – for example, those threatened with administration or which are experiencing other forms of financial distress (Barth, Beaver and Landsman, 1998). In such instances it is doubtful whether linear models of the relationship between the market value of equity and its determining variables can adequately capture the empirical relationships which exist in the area. [16]

## 5. Summary Conclusions

It is now some time since Burgstahler and Dichev (1997, p. 212) and Penman (2001, p. 692) observed that the theoretical basis for empirical work on the relationship between equity value and its determining variables is extremely weak. Unfortunately, their call for more refined theoretical modelling in the area has largely been ignored. Empirical work on the relationship between the market value of equity and its

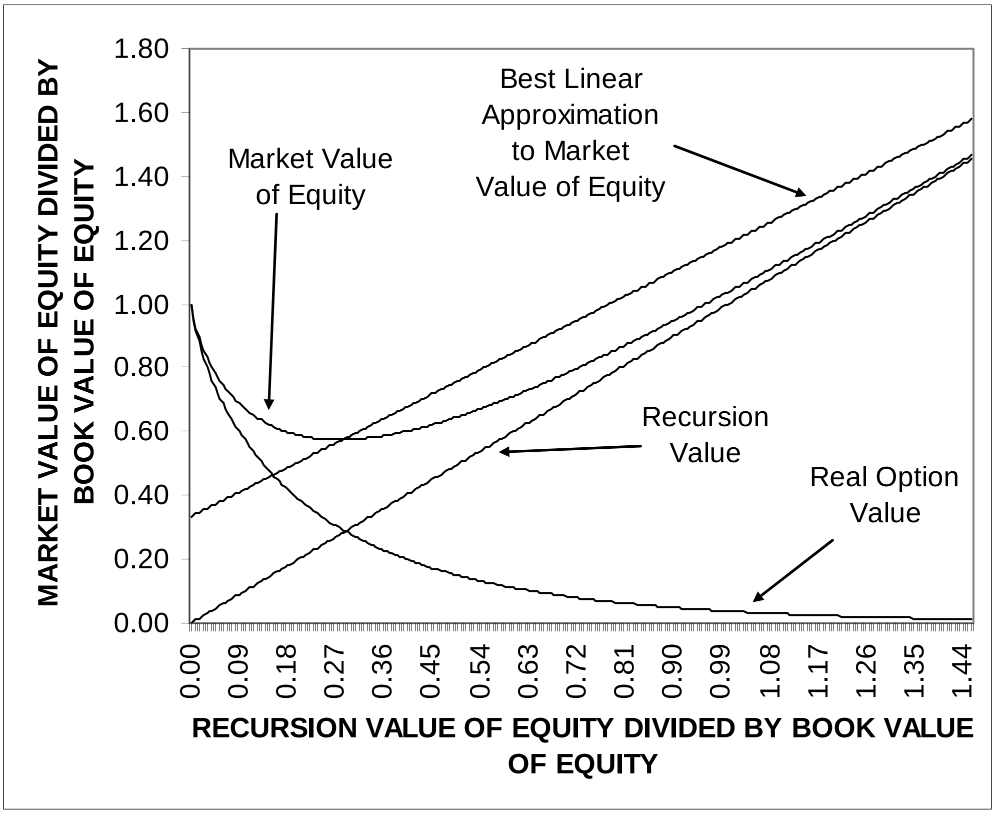
[16] Empiricists will often (implicitly) acknowledge the non-linear nature of the valuation relationships that exist in these situations by using dummy variables to allow regression coefficients to vary with so called “extreme” observations (Barth, Beaver and Landsman, 1998, pp. 5-9). However, whether this procedure can satisfactorily addresses the omitted variables problems implicit in their empirical work is yet to be demonstrated.

determining variables continues to be based on linear models that neglect the real option effects associated with a firm's ability to modify or even abandon its existing operating activities. It is well known, however, that real options induce a convex and potentially, highly non-linear relationship between equity values and their determining variables [Burgstahler and Dichev (1997), Ashton, Cooke and Tippett (2003)]. Given this, it is all but inevitable that when real options do impact on equity values systematic biases will arise in empirical work based on linear valuation models. Moreover our empirical analysis, which is based on U.K. data, confirms that there is a strong convex relationship between the market value of equity and earnings and that systematic biases do arise when linear approximations of the equity valuation relationship are mistakenly employed. These biases appear to be mainly confined to firms with relatively low earnings to book ratios or where the un-deflated book value of equity is comparatively small. Unfortunately, firms with these characteristics typically account for around 20% of the samples employed in empirical work (Burgsthaler and Dichev, 1997, p. 197; Ashton, Cooke and Tippett, 2003, p. 428) and so they can have a significant impact on parameter estimation.

Given the now extensive empirical evidence on this convexity issue it is again timely to renew the call for the development of more refined analytical models of the relationship between equity value and its determining variables. There are two areas in particular where the need for enhanced modelling procedures is urgent. First, relatively little is known about the impact that the real options available to firms have on the book values of assets and liabilities and the accounting policies implemented by firms. The few papers published on this topic [Gietzmann and Ostaszewski (1999, 2004), Ashton, Cooke, Tippett (2003), Ashton, Cooke, Tippett and Wang (2004)] have had relatively little impact on the empirical work conducted in the area - which remains largely "wedded" to linear valuation models that neglect the impact which real options can have on equity values. Second, even less is known about the appropriate econometric procedures to be used in empirical work in an environment where there is a non-linear relationship between equity values and their determining variables. Suffice it to say that if econometric procedures mistakenly assume the existence of a purely linear relationship between equity values and their determining variables, then it is all but inevitable there will be problems with omitted variables and scale effects in the data. Our analysis shows, however, that these problems can be mitigated by approximating the market value of equity in terms of a polynomial expression of its determining variables – although our evidence is also that the polynomial terms will have to be carried to a fairly high order if this technique is to be satisfactory. This, however, might present problems with relatively small samples where there are insufficient degrees of freedom to sustain the high order polynomial regression procedures envisaged by this procedure.

**FIGURE 1**

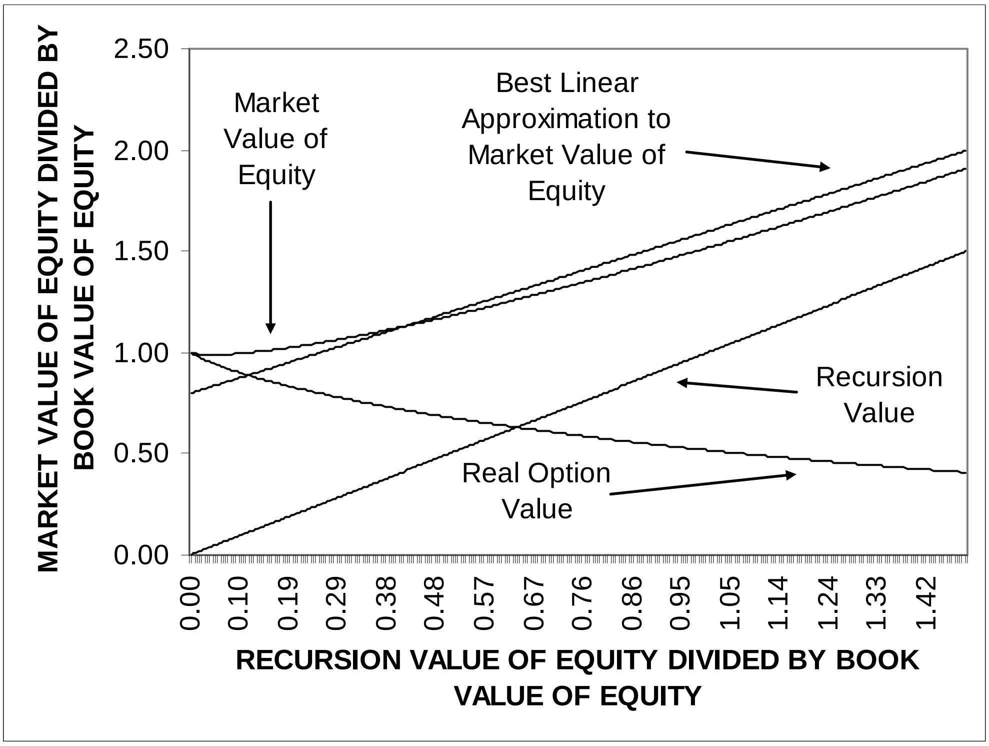
**PLOT OF RECURSION VALUE OF EQUITY, REAL OPTION VALUE OF EQUITY, OVERALL MARKET VALUE OF EQUITY AND LINEAR APPROXIMATION TO OVERALL VALUE OF EQUITY FOR A BRANCHING PROCESS ($\delta = \frac{1}{2}$) WITH RISK PARAMETER $\theta B = 2$.**




The upward sloping line emanating from the origin at a 45 degree angle is the normalised recursion value of equity, h. The downward sloping curve which asymptotes towards the recursion value axis is the normalised real option value of equity, $\frac{1}{2}\int_{-1}^{1}\exp\left(\frac{-4h}{1+z}\right)dz$. The sum of the normalised recursion and real option values is the total market value of equity divided by the book value of equity, $P(h,1,2,\frac{1}{2}) = h + \frac{1}{2}\int_{-1}^{1}\exp\left(\frac{-4h}{1+z}\right)dz$, and is represented by the convex curve which asymptotes towards the 45 degree line representing the normalised recursion value of equity. The line emanating from the point 0.3333 on the market value axis is the "best" linear approximation, $P(h,1,2,\frac{1}{2}) \approx 0.3333 + 0.8557h$ to the overall market value of equity.

**FIGURE 2**

**PLOT OF RECURSION VALUE OF EQUITY, REAL OPTION VALUE OF EQUITY, OVERALL MARKET VALUE OF EQUITY AND LINEAR APPROXIMATION TO OVERALL VALUE OF EQUITY FOR A BRANCHING PROCESS ($\delta = \frac{1}{2}$) WITH RISK PARAMETER $\theta B = 0.25$.**



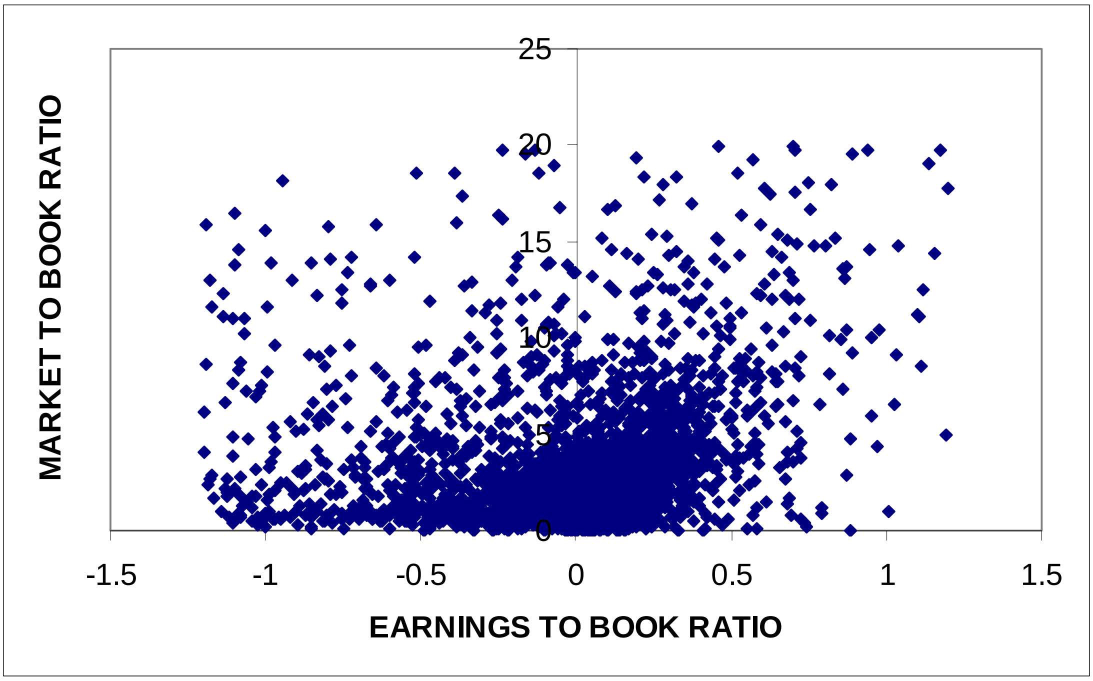

The upward sloping line emanating from the origin at a 45 degree angle is the normalised recursion value of equity, h. The downward sloping curve which asymptotes towards the recursion value axis is the normalised real option value of equity, $\frac{1}{2}\int_{-1}^{1}\exp(\frac{-0.5h}{1+z})dz$. The sum of the normalised recursion and real option values is the total market value of equity divided by the book value of equity, $P(h,1,0.25,\frac{1}{2}) = h + \frac{1}{2}\int_{-1}^{1}\exp(\frac{-0.5h}{1+z})dz$, and is represented by the convex curve which asymptotes towards the 45 degree line representing the normalised recursion value of equity. The line emanating from the point 0.8000 on the market value axis is the "best" linear approximation, $P(h,1,0.25,\frac{1}{2}) \approx 0.8000 + 0.7976h$ to the overall market value of equity.

**FIGURE 3**

**AGGREGATE DATA**

**MARKET VALUE OF EQUITY DIVIDED BY THE BOOK VALUE OF EQUITY AGAINST EARNINGS DIVIDED BY THE BOOK VALUE OF EQUITY**

**N = 4,043 FIRM-YEARS COVERING THE PERIOD FROM 2001 UNTIL 2004**



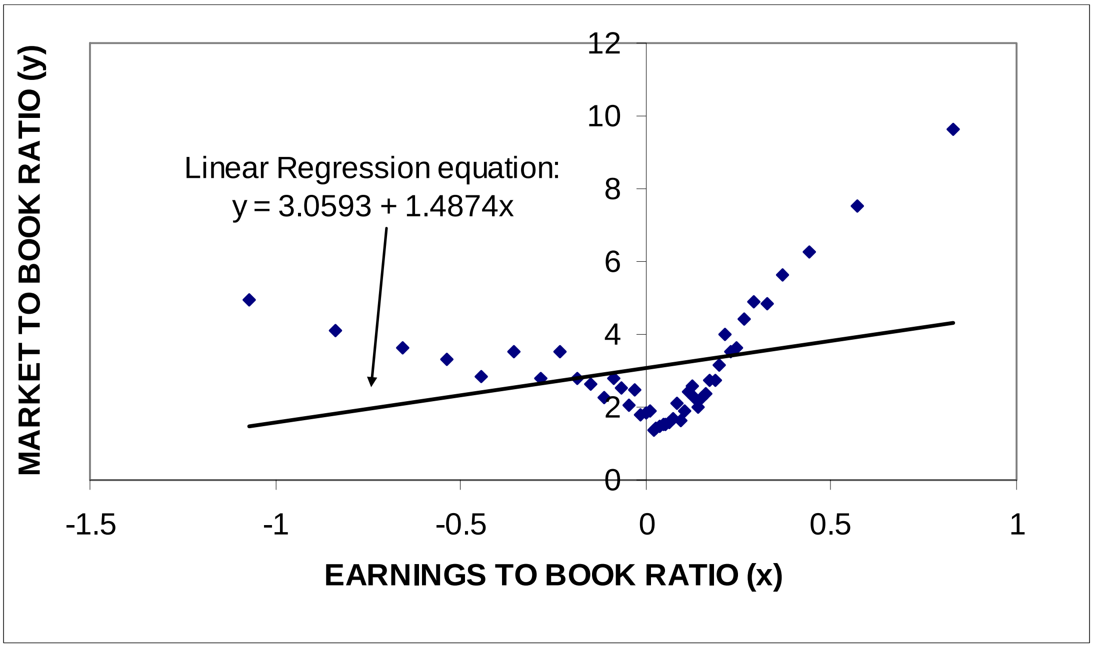

This graph summarises the relationship between the market value of equity divided by the book value of equity and the earnings attributable to equity divided by the book value of equity for U.K. firms from all industrial classifications over the period from 2001 to 2004. The graph is based on 4,043 firm-years of data. The market value of equity for a particular firm is defined as its share price multiplied by the number of shares on issue 100 days after its balance sheet date. Earnings is the firm's net income before exceptional items less any preference dividends over the year covered by its balance sheet date. The book value of equity is defined as the net tangible assets attributable to equity less the book value of intangible assets (both at the beginning of the year covered by the firm's balance sheet date). The graph excludes some "extreme" observations in order to prevent "squashing" or large gaps appearing between the inner and outer data points appearing on the Figure.

## FIGURE 4

## AGGREGATE DATA

## ORDERED SUB-GROUP AVERAGES OF RATIO OF MARKET VALUE OF EQUITY TO BOOK VALUE OF EQUITY AGAINST RATIO OF EARNINGS TO BOOK VALUE OF EQUITY

## N = 4,043 FIRM-YEARS COVERING THE PERIOD FROM 2001 UNTIL 2004



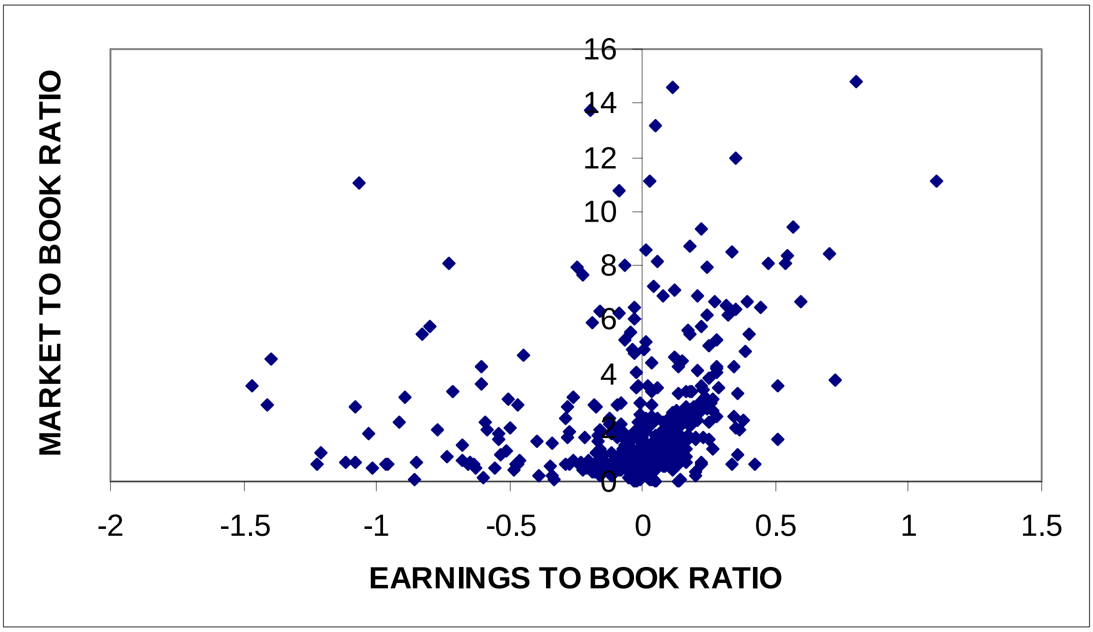

For this graph the 4,043 earnings to book ratios on which Figure 3 is based were sorted from the smallest to the largest ratios. The simple average of the lowest 75 earnings to book ratios was then calculated together with the simple average of their market to book ratios. The average of the next 75 lowest earnings to book ratios was then calculated together with the average of their market to book ratios. Continuing this procedure across the entire sample gives the averaged ordered pairs plotted in the above graph. The Linear Regression equation is obtained by running a simple O.L.S regression between the sample of ordered average earnings to book ratios and their corresponding average market to book ratios. The graph excludes some “extreme” observations in order to prevent “squashing” or large gaps appearing between the inner and outer data points appearing on the Figure.

**FIGURE 5**

**FINANCIAL SERVICES**

**MARKET VALUE OF EQUITY DIVIDED BY THE BOOK VALUE OF EQUITY AGAINST EARNINGS DIVIDED BY THE BOOK VALUE OF EQUITY**

**N = 571 FIRM-YEARS COVERING THE PERIOD FROM 2001 UNTIL 2004**



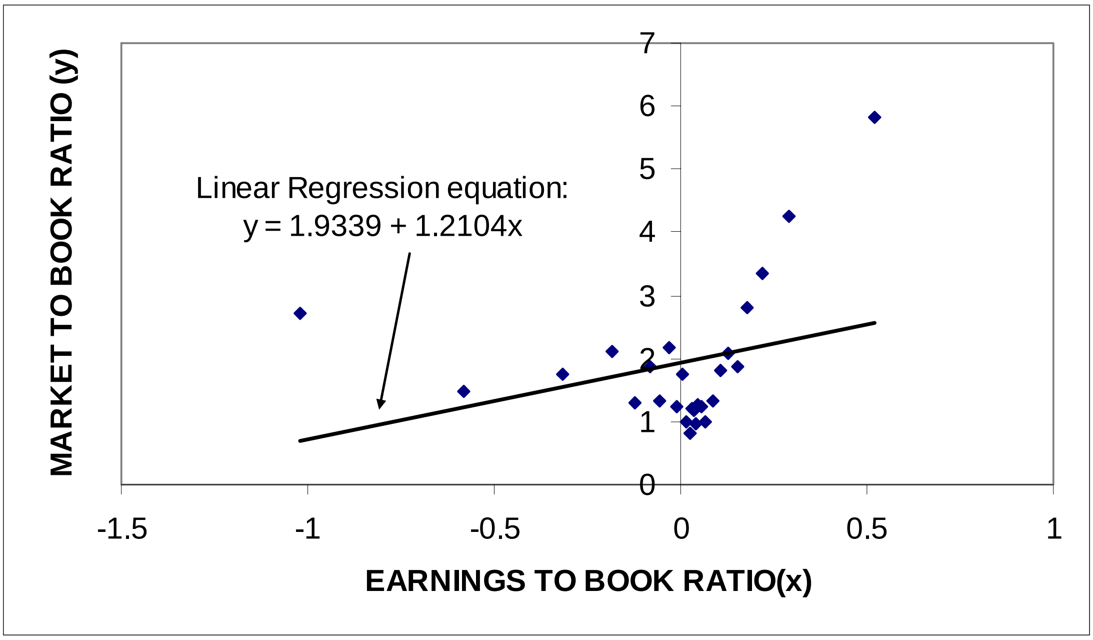

This graph summarises the relationship between the market value of equity divided by the book value of equity and the earnings attributable to equity divided by the book value of equity for U.K. firms from the Financial Services industrial classification over the period from 2001 to 2004. The graph is based on 571 firm-years of data. The market value of equity for a particular firm is defined as its share price multiplied by the number of shares on issue 100 days after its balance sheet date. Earnings is the firm's net income before exceptional items less any preference dividends over the year covered by its balance sheet date. Finally, the book value of equity is defined as the net tangible assets attributable to equity less the book value of intangible assets (both at the beginning of the year covered by the firm's balance sheet date).

**FIGURE 6**

**FINANCIAL SERVICES**

**ORDERED SUB-GROUP AVERAGES OF RATIO OF MARKET VALUE OF EQUITY TO BOOK VALUE OF EQUITY AGAINST RATIO OF EARNINGS TO BOOK VALUE OF EQUITY**

**N = 571 FIRM-YEARS COVERING THE PERIOD FROM 2001 UNTIL 2004**



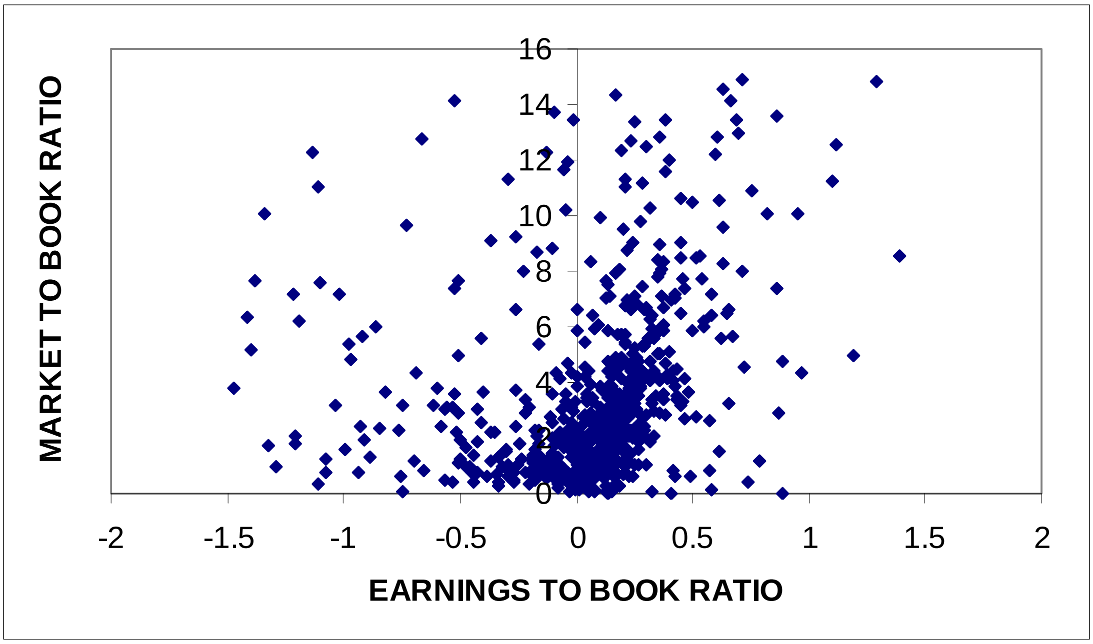

For this graph the 571 earnings to book ratios on which Figure 5 is based were sorted from the smallest to the largest ratios. The simple average of the lowest 22 earnings to book ratios was then calculated together with the simple average of their market to book ratios. The average of the next 22 lowest earnings to book ratios was then calculated together with the average of their market to book ratios. This procedure was continued until there were 26 ordered average earnings to book ratios together with their average market to book ratios and all 26 order pairs of these averages are plotted in the above graph. The Linear Regression equation is obtained by running a simple O.L.S regression between the 26 ordered average earnings to book ratios and their corresponding average market to book ratios.

**FIGURE 7**

**INDUSTRIAL GOODS AND SERVICES**

**MARKET VALUE OF EQUITY DIVIDED BY THE BOOK VALUE OF EQUITY AGAINST EARNINGS DIVIDED BY THE BOOK VALUE OF EQUITY**

**N = 758 FIRM-YEARS COVERING THE PERIOD FROM 2001 UNTIL 2004**



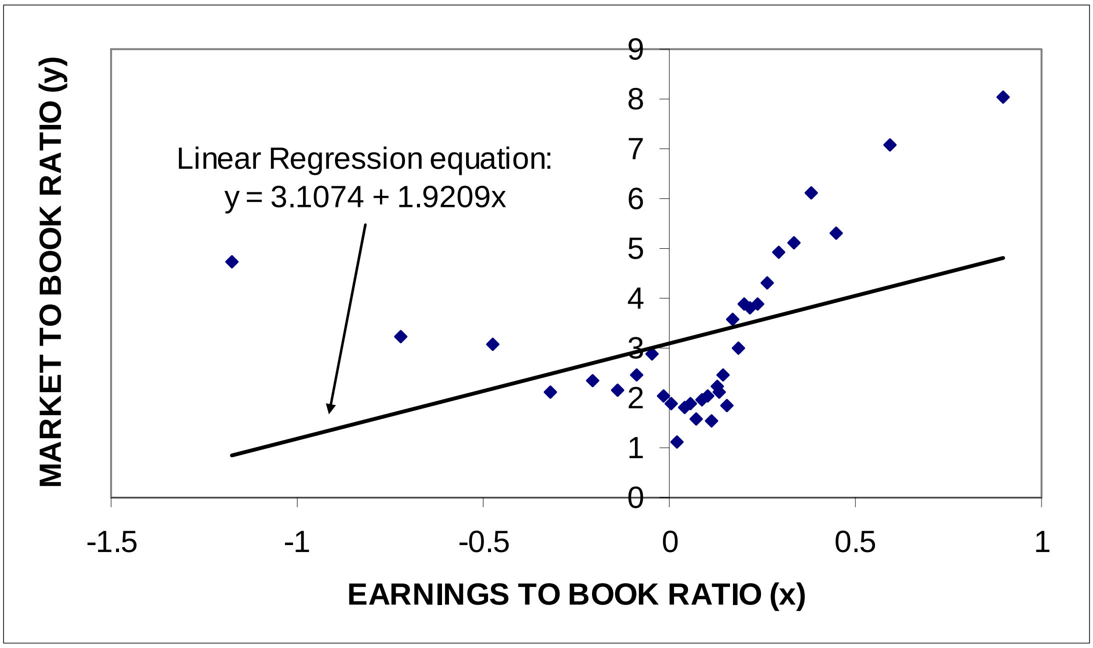

This graph summarises the relationship between the market value of equity divided by the book value of equity and the earnings attributable to equity divided by the book value of equity for U.K. firms from the Industrial Goods and Services industrial classification over the period from 2001 to 2004. The graph is based on 758 firm-years of data. The market value of equity for a particular firm is defined as its share price multiplied by the number of shares on issue 100 days after its balance sheet date. Earnings is the firm's net income before exceptional items less any preference dividends over the year covered by its balance sheet date. Finally, the book value of equity is defined as the net tangible assets attributable to equity less the book value of intangible assets (both at the beginning of the year covered by the firm's balance sheet date).

**FIGURE 8**

**INDUSTRIAL GOODS AND SERVICES**

**ORDERED SUB-GROUP AVERAGES OF RATIO OF MARKET VALUE OF EQUITY TO BOOK VALUE OF EQUITY AGAINST RATIO OF EARNINGS TO BOOK VALUE OF EQUITY**

**N = 758 FIRM-YEARS COVERING THE PERIOD FROM 2001 UNTIL 2004**



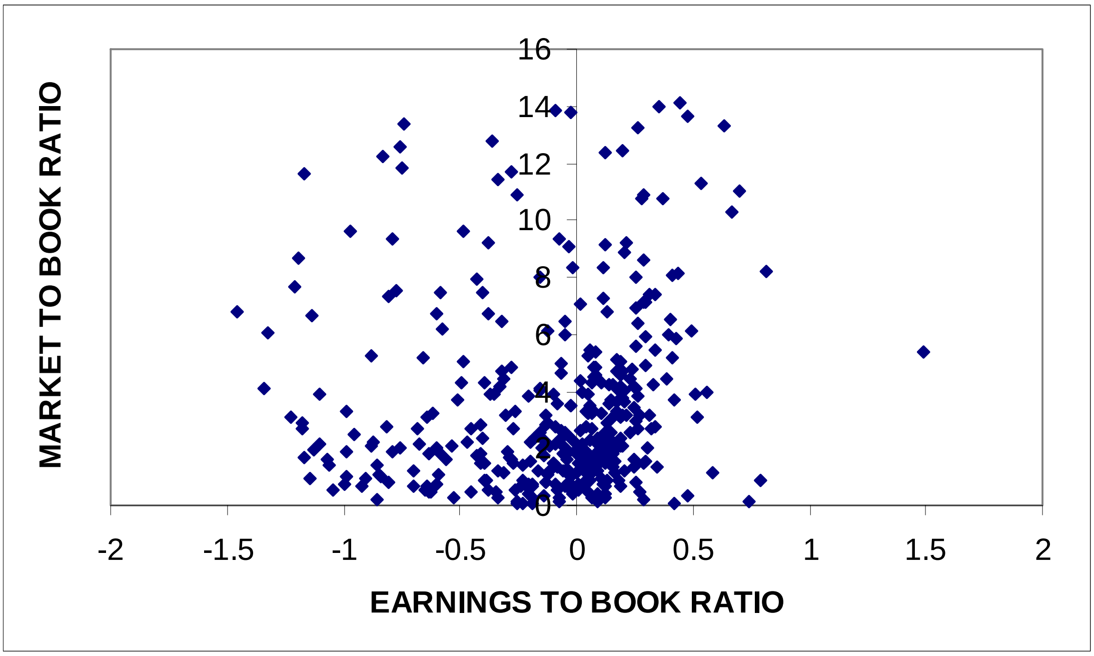

For this graph the 758 earnings to book ratios on which Figure 7 is based were sorted from the smallest to the largest ratios. The simple average of the lowest 23 earnings to book ratios was then calculated together with the simple average of their market to book ratios. The average of the next 23 lowest earnings to book ratios was then calculated together with the average of their market to book ratios. This procedure was continued until there were 33 ordered average earnings to book ratios together with their average market to book ratios and all 33 order pairs of these averages are plotted in the above graph. The Linear Regression equation is obtained by running a simple O.L.S regression between the 33 ordered average earnings to book ratios and their corresponding average market to book ratios.

**FIGURE 9**

**TECHNOLOGY**

**MARKET VALUE OF EQUITY DIVIDED BY THE BOOK VALUE OF EQUITY AGAINST EARNINGS DIVIDED BY THE BOOK VALUE OF EQUITY**

**N = 384 FIRM-YEARS COVERING THE PERIOD FROM 2001 UNTIL 2004**



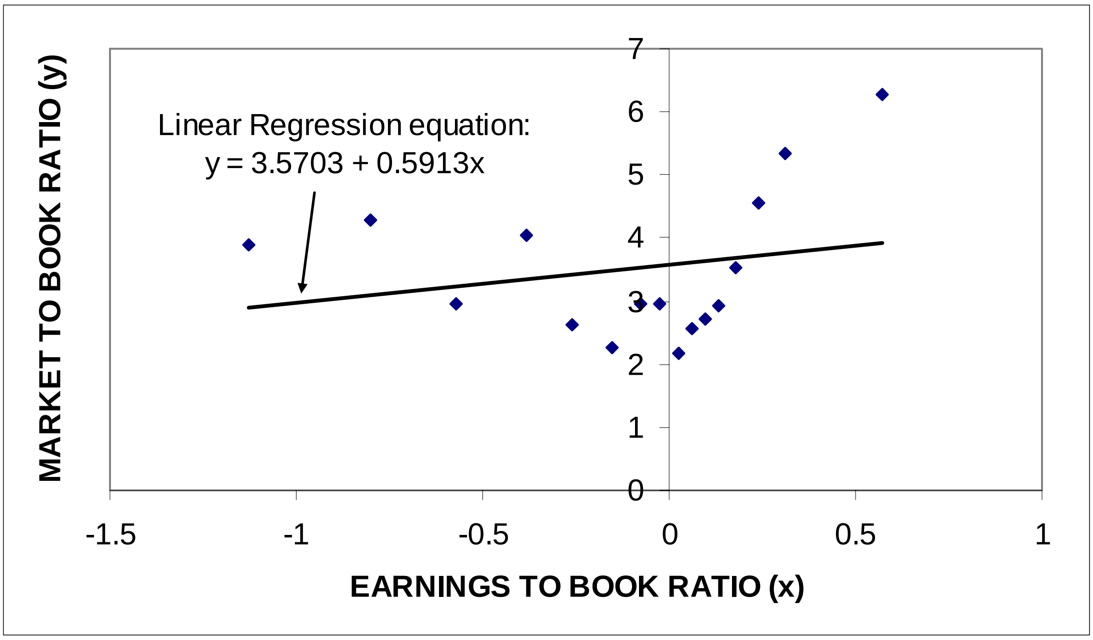

This graph summarises the relationship between the market value of equity divided by the book value of equity and the earnings attributable to equity divided by the book value of equity for U.K. firms from the Technology industrial classification over the period from 2001 to 2004. The graph is based on 384 firm-years of data. The market value of equity for a particular firm is defined as its share price multiplied by the number of shares on issue 100 days after its balance sheet date. Earnings is the firm's net income before exceptional items less any preference dividends over the year covered by its balance sheet date. Finally, the book value of equity is defined as the net tangible assets attributable to equity less the book value of intangible assets (both at the beginning of the year covered by the firm's balance sheet date).

**FIGURE 10**

**TECHNOLOGY**

**ORDERED SUB-GROUP AVERAGES OF RATIO OF MARKET VALUE OF EQUITY TO BOOK VALUE OF EQUITY AGAINST RATIO OF EARNINGS TO BOOK VALUE OF EQUITY**

**N = 384 FIRM-YEARS COVERING THE PERIOD FROM 2001 UNTIL 2004**



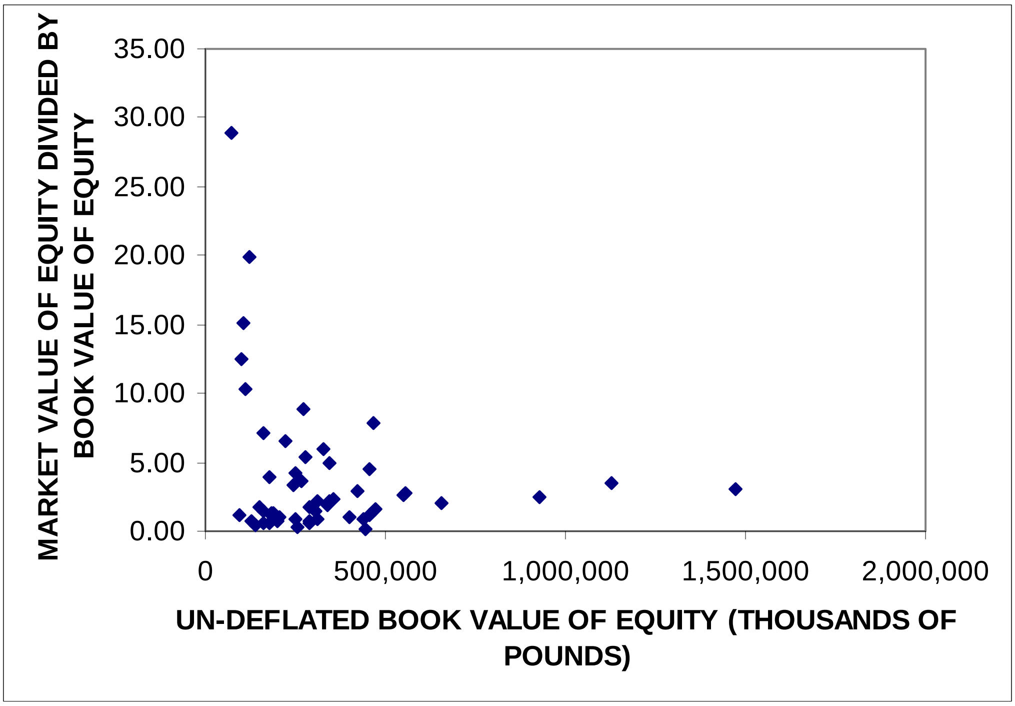

For this graph the 384 earnings to book ratios on which Figure 9 is based were sorted from the smallest to the largest ratios. The simple average of the lowest 24 earnings to book ratios was then calculated together with the simple average of their market to book ratios. The average of the next 24 lowest earnings to book ratios was then calculated together with the average of their market to book ratios. This procedure was continued until there were 16 ordered average earnings to book ratios together with their average market to book ratios and all 16 order pairs of these averages are plotted in the above graph. The Linear Regression equation is obtained by running a simple O.L.S regression between the 16 ordered average earnings to book ratios and their corresponding average market to book ratios.

**FIGURE 11**

**AGGREGATE DATA**

**ORDERED SUB-GROUP AVERAGES OF RATIO OF MARKET VALUE OF EQUITY DIVIDED BY BOOK VALUE OF EQUITY AGAINST THE UN-DEFLATED BOOK VALUE OF EQUITY**

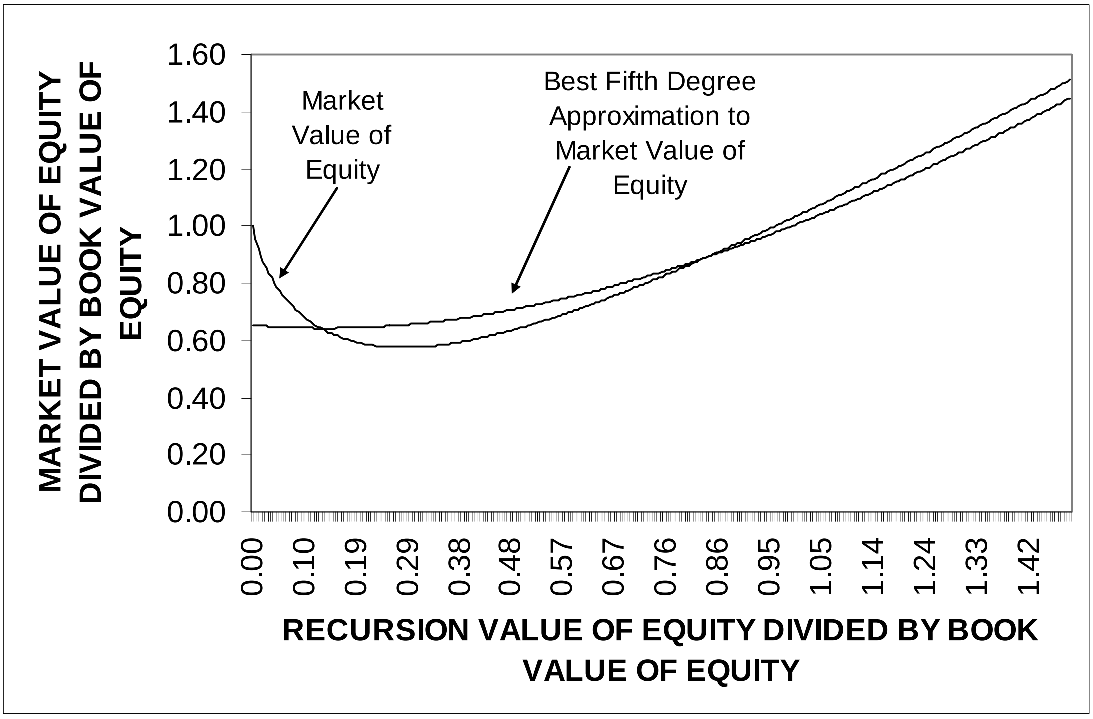
**N = 4,043 FIRM-YEARS COVERING THE PERIOD FROM 2001 UNTIL 2004**

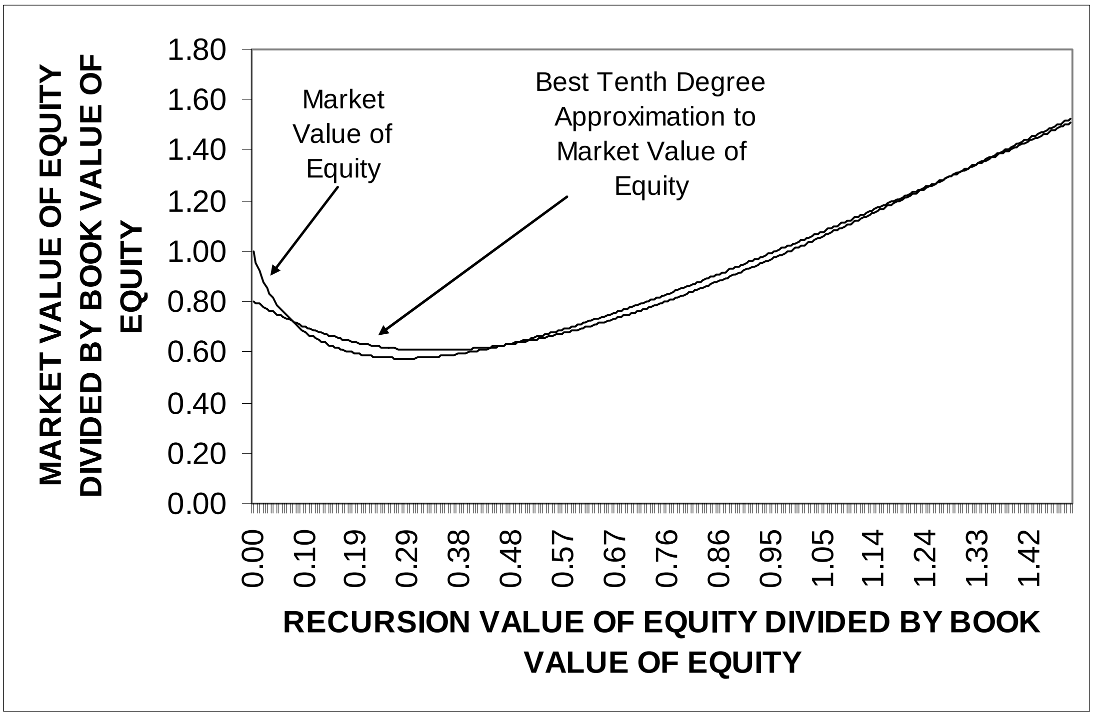


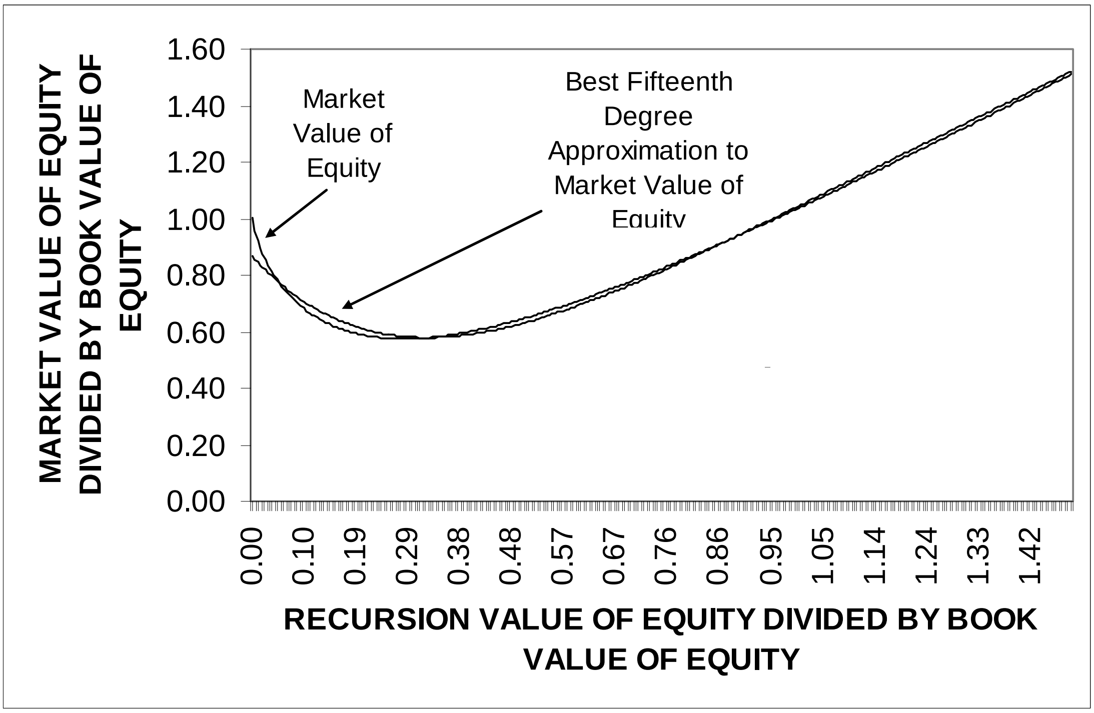

This graph summarises the relationship between the market to book ratio and the un-deflated book value of equity for U.K. firms from all industrial classifications over the period from 2001 to 2004. The graph is based on 4,043 firm-years of data. The simple average of the lowest 75 market to book ratios was calculated together with the simple average of their un-deflated book values of equity. The average of the next 75 lowest market to book ratios was then calculated together with the average of their un-deflated book values of equity. Continuing this procedure across the entire sample gives the averaged ordered pairs plotted in the above graph. The graph excludes some "extreme" observations in order to prevent "squashing" or large gaps appearing between the inner and outer data points appearing on the Figure.

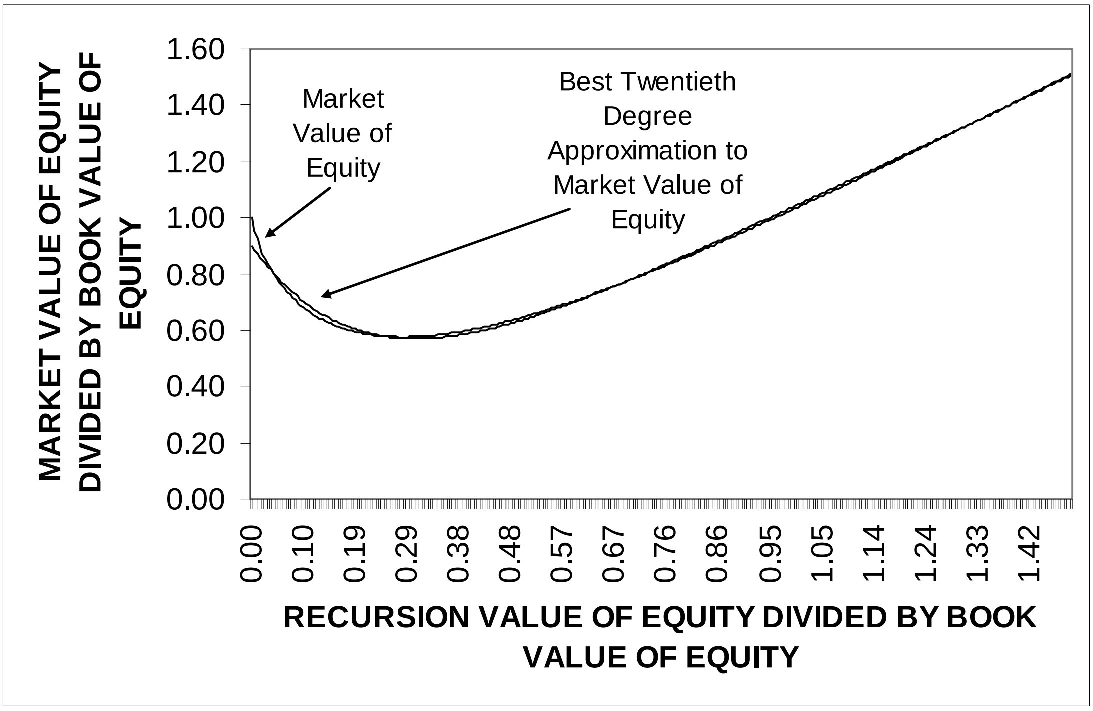
**FIGURE 12**

**PLOT OF POLYNOMIAL APPROXIMATIONS TO OVERALL MARKET VALUE OF EQUITY FOR BRANCHING PROCESS ($\delta = \frac{1}{2}$) WITH RISK PARAMETER θB = 2.**






The two curves in the above graphs are firstly, the market value of equity divided by the book value of equity, $P(h,1,2,\frac{1}{2}) = h + \frac{1}{2}\int_{-1}^{1} \exp(\frac{-4h}{1+z})dz$. The second curve is the Laguerre polynomial approximation $P(h,1,2,\frac{1}{2}) \approx \sum_{n=0}^{m} \alpha_n L_n(h)$ for m = 5 (first graph), m = 10 (second graph), m = 15 (third graph) and m = 20 (fourth graph).

## TABLE 1

## PROPORTION OF SQUARED VARIATION IN EQUITY VALUATION FUNCTION ASSOCIATED WITH LAGUERRE POLYNOMIALS

| | ← $\theta B = 0.25$ → | | | ← $\theta B = 0.50$ → | | | ← $\theta B = 0.75$ → | | | ← $\theta B = 1.00$ → | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| m | $\alpha_m$ | $\alpha_m^2 \cdot [\sum_{n=0}^{\infty} \alpha_n^2)]^{-1}$ | $R_m^2$ | $\alpha_m$ | $\alpha_m^2 \cdot [\sum_{n=0}^{\infty} \alpha_n^2)]^{-1}$ | $R_m^2$ | $\alpha_m$ | $\alpha_m^2 \cdot [\sum_{n=0}^{\infty} \alpha_n^2)]^{-1}$ | $R_m^2$ | $\alpha_m$ | $\alpha_m^2 \cdot [\sum_{n=0}^{\infty} \alpha_n^2)]^{-1}$ | $R_m^2$ |
| 0 | 1.5976 | 0.798338 | 0.798338 | 1.4507 | 0.768643 | 0.768643 | 1.3645 | 0.739860 | 0.739860 | 1.3069 | 0.714979 | 0.714979 |
| 1 | -0.7976 | 0.198996 | 0.997334 | -0.7840 | 0.224510 | 0.993153 | -0.7931 | 0.249942 | 0.989802 | -0.8069 | 0.272539 | 0.987518 |
| 2 | 0.0800 | 0.002002 | 0.999336 | 0.1111 | 0.004509 | 0.997662 | 0.1224 | 0.005958 | 0.995760 | 0.1250 | 0.006541 | 0.994059 |
| 3 | 0.0373 | 0.000436 | 0.999772 | 0.0617 | 0.001392 | 0.999054 | 0.0758 | 0.002283 | 0.998044 | 0.0833 | 0.002907 | 0.996966 |
| 4 | 0.0203 | 0.000128 | 0.999900 | 0.0370 | 0.000501 | 0.999555 | 0.0491 | 0.000960 | 0.999003 | 0.0573 | 0.001374 | 0.998341 |
| 5 | 0.0124 | 0.000048 | 0.999948 | 0.0239 | 0.000208 | 0.999763 | 0.0333 | 0.000442 | 0.999445 | 0.0406 | 0.000691 | 0.999031 |
| 6 | 0.0083 | 0.000022 | 0.999970 | 0.0164 | 0.000098 | 0.999861 | 0.0236 | 0.000221 | 0.999667 | 0.0297 | 0.000369 | 0.999400 |
| 7 | 0.0060 | 0.000011 | 0.999981 | 0.0118 | 0.000051 | 0.999912 | 0.0174 | 0.000120 | 0.999786 | 0.0223 | 0.000209 | 0.999609 |
| 8 | 0.0045 | 0.000006 | 0.999987 | 0.0089 | 0.000029 | 0.999941 | 0.0132 | 0.000069 | 0.999856 | 0.0172 | 0.000124 | 0.999733 |
| 9 | 0.0035 | 0.000004 | 0.999991 | 0.0069 | 0.000018 | 0.999958 | 0.0104 | 0.000043 | 0.999898 | 0.0136 | 0.000078 | 0.999811 |
| 10 | 0.0028 | 0.000002 | 0.999994 | 0.0056 | 0.000011 | 0.999970 | 0.0083 | 0.000027 | 0.999926 | 0.0110 | 0.000051 | 0.999862 |
| 15 | 0.0012 | 0.000000 | 0.999998 | 0.0024 | 0.000002 | 0.999991 | 0.0036 | 0.000005 | 0.999978 | 0.0048 | 0.000009 | 0.999959 |
| 30 | 0.0003 | 0.000000 | 1.000000 | 0.0006 | 0.000000 | 0.999999 | 0.0009 | 0.000000 | 0.999997 | 0.0011 | 0.000001 | 0.999995 |
| 50 | 0.0001 | 0.000000 | 1.000000 | 0.0002 | 0.000000 | 1.000000 | 0.0003 | 0.000000 | 0.999999 | 0.0004 | 0.000000 | 0.999999 |
| 75 | 0.0000 | 0.000000 | 1.000000 | 0.0001 | 0.000000 | 1.000000 | 0.0001 | 0.000000 | 1.000000 | 0.0002 | 0.000000 | 1.000000 |
| 100 | 0.0000 | 0.000000 | 1.000000 | 0.0001 | 0.000000 | 1.000000 | 0.0001 | 0.000000 | 1.000000 | 0.0001 | 0.000000 | 1.000000 |

| | ← θB = 2 → | | | ← θB = 4 → | | | ← θB = 6 → | | | ← θB = 8 → | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| m | $\alpha_m$ | $\alpha_m^2.[\sum_{n=0}^{\infty}\alpha_n^2]^{-1}$ | $R_m^2$ | $\alpha_m$ | $\alpha_m^2.[\sum_{n=0}^{\infty}\alpha_n^2]^{-1}$ | $R_m^2$ | $\alpha_m$ | $\alpha_m^2.[\sum_{n=0}^{\infty}\alpha_n^2]^{-1}$ | $R_m^2$ | $\alpha_m$ | $\alpha_m^2.[\sum_{n=0}^{\infty}\alpha_n^2]^{-1}$ | $R_m^2$ |
| 0 | 1.1891 | 0.648684 | 0.648684 | 1.1074 | 0.590307 | 0.590307 | 1.0751 | 0.564577 | 0.564577 | 1.0577 | 0.550209 | 0.550209 |
| 1 | -0.8557 | 0.335968 | 0.984652 | -0.9074 | 0.396343 | 0.986650 | -0.9322 | 0.424505 | 0.989082 | -0.9466 | 0.440686 | 0.990896 |
| 2 | 0.1111 | 0.005664 | 0.990316 | 0.0800 | 0.003081 | 0.989731 | 0.0612 | 0.001831 | 0.990913 | 0.0494 | 0.001199 | 0.992095 |
| 3 | 0.0864 | 0.003426 | 0.993743 | 0.0693 | 0.002314 | 0.992044 | 0.0554 | 0.001499 | 0.992412 | 0.0457 | 0.001028 | 0.993123 |
| 4 | 0.0679 | 0.002115 | 0.995858 | 0.0603 | 0.001748 | 0.993793 | 0.0502 | 0.001230 | 0.993642 | 0.0424 | 0.000883 | 0.994006 |
| 5 | 0.0539 | 0.001333 | 0.997191 | 0.0525 | 0.001329 | 0.995122 | 0.0455 | 0.001013 | 0.994655 | 0.0393 | 0.000759 | 0.994765 |
| 6 | 0.0433 | 0.000858 | 0.998050 | 0.0460 | 0.001016 | 0.996138 | 0.0414 | 0.000836 | 0.995491 | 0.0365 | 0.000654 | 0.995420 |
| 7 | 0.0351 | 0.000565 | 0.998614 | 0.0403 | 0.000782 | 0.996920 | 0.0376 | 0.000692 | 0.996183 | 0.0339 | 0.000565 | 0.995984 |
| 8 | 0.0287 | 0.000379 | 0.998994 | 0.0355 | 0.000606 | 0.997526 | 0.0343 | 0.000575 | 0.996758 | 0.0315 | 0.000488 | 0.996472 |
| 9 | 0.0238 | 0.000260 | 0.999253 | 0.0313 | 0.000472 | 0.997998 | 0.0313 | 0.000479 | 0.997236 | 0.0293 | 0.000423 | 0.996895 |
| 10 | 0.0199 | 0.000182 | 0.999435 | 0.0277 | 0.000370 | 0.998369 | 0.0286 | 0.000400 | 0.997636 | 0.0273 | 0.000367 | 0.997261 |
| 15 | 0.0093 | 0.000040 | 0.999821 | 0.0159 | 0.000121 | 0.999350 | 0.0187 | 0.000170 | 0.998861 | 0.0194 | 0.000185 | 0.998494 |
| 30 | 0.0023 | 0.000002 | 0.999978 | 0.0045 | 0.000010 | 0.999908 | 0.0065 | 0.000021 | 0.999798 | 0.0079 | 0.000031 | 0.999669 |
| 50 | 0.0008 | 0.000000 | 0.999996 | 0.0016 | 0.000001 | 0.999982 | 0.0024 | 0.000003 | 0.999959 | 0.0032 | 0.000005 | 0.999928 |
| 75 | 0.0004 | 0.000000 | 0.999999 | 0.0007 | 0.000000 | 0.999996 | 0.0011 | 0.000001 | 0.999992 | 0.0014 | 0.000001 | 0.999986 |
| 100 | 0.0002 | 0.000000 | 1.000000 | 0.0004 | 0.000000 | 1.000000 | 0.0006 | 0.000000 | 1.000000 | 0.0008 | 0.000000 | 1.000000 |

The above Table gives the mth order Laguerre coefficient, $\alpha_m$, and the proportion, $\alpha_m^2.[\sum_{n=0}^{\infty}\alpha_n^2]^{-1}$, of the squared variation in the equity valuation function, $P(h,1,B\theta,\frac{1}{2})$, which is accounted for by the given Laguerre polynomial. It also gives the pseudo $R_m^2$ which summarises the accumulated proportion of the squared variation in the equity valuation function which is accounted for by the first m Laguerre polynomials.

## TABLE 2

## TABULAR VALUES OF THE REAL OPTION INTEGRAL WHEN $\delta = \frac{1}{2}$

$$R(y) = \frac{1}{2}\int_{-1}^{1} \exp\left(\frac{-2y}{1+z}\right)dz$$

| y | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | 1.0000 | 0.9492 | 0.9128 | 0.8817 | 0.8537 | 0.8280 | 0.8041 | 0.7819 | 0.7610 | 0.7412 |
| **0.1** | 0.7225 | 0.7047 | 0.6877 | 0.6715 | 0.6560 | 0.6410 | 0.6267 | 0.6128 | 0.5995 | 0.5866 |
| **0.2** | 0.5742 | 0.5622 | 0.5505 | 0.5393 | 0.5283 | 0.5177 | 0.5074 | 0.4975 | 0.4877 | 0.4783 |
| **0.3** | 0.4691 | 0.4602 | 0.4515 | 0.4430 | 0.4348 | 0.4267 | 0.4189 | 0.4112 | 0.4038 | 0.3965 |
| **0.4** | 0.3894 | 0.3824 | 0.3756 | 0.3690 | 0.3626 | 0.3562 | 0.3500 | 0.3440 | 0.3381 | 0.3323 |
| **0.5** | 0.3266 | 0.3211 | 0.3157 | 0.3104 | 0.3052 | 0.3001 | 0.2951 | 0.2902 | 0.2855 | 0.2808 |
| **0.6** | 0.2762 | 0.2717 | 0.2673 | 0.2630 | 0.2587 | 0.2546 | 0.2505 | 0.2465 | 0.2426 | 0.2387 |
| **0.7** | 0.2349 | 0.2312 | 0.2276 | 0.2240 | 0.2205 | 0.2171 | 0.2137 | 0.2104 | 0.2072 | 0.2040 |
| **0.8** | 0.2009 | 0.1978 | 0.1948 | 0.1918 | 0.1889 | 0.1860 | 0.1832 | 0.1804 | 0.1777 | 0.1750 |
| **0.9** | 0.1724 | 0.1698 | 0.1673 | 0.1648 | 0.1623 | 0.1599 | 0.1576 | 0.1552 | 0.1530 | 0.1507 |
| **1.0** | 0.1485 | 0.1463 | 0.1442 | 0.1421 | 0.1400 | 0.1380 | 0.1360 | 0.1340 | 0.1321 | 0.1302 |
| **1.1** | 0.1283 | 0.1264 | 0.1246 | 0.1228 | 0.1211 | 0.1193 | 0.1176 | 0.1160 | 0.1143 | 0.1127 |
| **1.2** | 0.1111 | 0.1095 | 0.1080 | 0.1065 | 0.1050 | 0.1035 | 0.1020 | 0.1006 | 0.0992 | 0.0978 |
| **1.3** | 0.0964 | 0.0951 | 0.0938 | 0.0925 | 0.0912 | 0.0899 | 0.0887 | 0.0875 | 0.0862 | 0.0851 |
| **1.4** | 0.0839 | 0.0827 | 0.0816 | 0.0805 | 0.0794 | 0.0783 | 0.0772 | 0.0762 | 0.0751 | 0.0741 |
| **1.5** | 0.0731 | 0.0721 | 0.0711 | 0.0702 | 0.0692 | 0.0683 | 0.0674 | 0.0665 | 0.0656 | 0.0647 |
| **1.6** | 0.0638 | 0.0629 | 0.0621 | 0.0613 | 0.0604 | 0.0596 | 0.0588 | 0.0581 | 0.0573 | 0.0565 |
| **1.7** | 0.0558 | 0.0550 | 0.0543 | 0.0536 | 0.0529 | 0.0522 | 0.0515 | 0.0508 | 0.0501 | 0.0495 |
| **1.8** | 0.0488 | 0.0482 | 0.0475 | 0.0469 | 0.0463 | 0.0457 | 0.0451 | 0.0445 | 0.0439 | 0.0433 |
| **1.9** | 0.0428 | 0.0422 | 0.0417 | 0.0411 | 0.0406 | 0.0401 | 0.0395 | 0.0390 | 0.0385 | 0.0380 |
| **2.0** | 0.0375 | 0.0370 | 0.0366 | 0.0361 | 0.0356 | 0.0352 | 0.0347 | 0.0343 | 0.0338 | 0.0334 |
| **2.1** | 0.0330 | 0.0325 | 0.0321 | 0.0317 | 0.0313 | 0.0309 | 0.0305 | 0.0301 | 0.0297 | 0.0294 |
| **2.2** | 0.0290 | 0.0286 | 0.0282 | 0.0279 | 0.0275 | 0.0272 | 0.0268 | 0.0265 | 0.0262 | 0.0258 |
| **2.3** | 0.0255 | 0.0252 | 0.0249 | 0.0245 | 0.0242 | 0.0239 | 0.0236 | 0.0233 | 0.0230 | 0.0227 |
| **2.4** | 0.0225 | 0.0222 | 0.0219 | 0.0216 | 0.0214 | 0.0211 | 0.0208 | 0.0206 | 0.0203 | 0.0200 |
| **2.5** | 0.0198 | 0.0196 | 0.0193 | 0.0191 | 0.0188 | 0.0186 | 0.0184 | 0.0181 | 0.0179 | 0.0177 |
| **2.6** | 0.0175 | 0.0172 | 0.0170 | 0.0168 | 0.0166 | 0.0164 | 0.0162 | 0.0160 | 0.0158 | 0.0156 |
| **2.7** | 0.0154 | 0.0152 | 0.0150 | 0.0149 | 0.0147 | 0.0145 | 0.0143 | 0.0141 | 0.0140 | 0.0138 |
| **2.8** | 0.0136 | 0.0134 | 0.0133 | 0.0131 | 0.0130 | 0.0128 | 0.0126 | 0.0125 | 0.0123 | 0.0122 |
| **2.9** | 0.0120 | 0.0119 | 0.0117 | 0.0116 | 0.0115 | 0.0113 | 0.0112 | 0.0110 | 0.0109 | 0.0108 |
| **3.0** | 0.0106 | 0.0105 | 0.0104 | 0.0103 | 0.0101 | 0.0100 | 0.0099 | 0.0098 | 0.0096 | 0.0095 |
| **3.1** | 0.0094 | 0.0093 | 0.0092 | 0.0091 | 0.0090 | 0.0089 | 0.0088 | 0.0086 | 0.0085 | 0.0084 |
| **3.2** | 0.0083 | 0.0082 | 0.0081 | 0.0080 | 0.0079 | 0.0078 | 0.0078 | 0.0077 | 0.0076 | 0.0075 |
| **3.3** | 0.0074 | 0.0073 | 0.0072 | 0.0071 | 0.0070 | 0.0070 | 0.0069 | 0.0068 | 0.0067 | 0.0066 |
| **3.4** | 0.0065 | 0.0065 | 0.0064 | 0.0063 | 0.0062 | 0.0062 | 0.0061 | 0.0060 | 0.0059 | 0.0059 |
| **3.5** | 0.0058 | 0.0057 | 0.0057 | 0.0056 | 0.0055 | 0.0055 | 0.0054 | 0.0053 | 0.0053 | 0.0052 |

# APPENDIX

## Averaging "Out" the Information Variable in the Ohlson (1995) Model

From equation (2) of the text, recall that under the Ohlson (1995, pp. 667-669) model the recursion value of equity, $\eta(t)$, is related to the book value of equity, $b(t)$, earnings, $x(t)$, and the information variable, $\nu(t)$, in accordance with the formula:

$$\eta(t) = c_1 b(t) + c_2 x(t) + c_3 \nu(t) \qquad \text{(A1)}$$

where $c_1$, $c_2$ and $c_3$ are the relevant valuation coefficients. Moreover, in a continuous time context the information variable evolves in accordance with the differential equation:[17]

$$\nu'(t) = -\gamma \nu(t) + z'(t) \qquad \text{(A2)}$$

where $\gamma$ is a speed of adjustment coefficient and $z'(t)$ is a Wiener process with variance parameter $\sigma^2$. The unique solution to this differential equation is [Hoel, Port and Stone (1972, pp. 154-157)]:

$$\nu(t) = \nu(0)e^{-\gamma t} + \int_0^t e^{-\gamma(t - s)} dz(s) \qquad \text{(A3)}$$

This in turn implies that the information variable has a mean of $E_0[\nu(t)] = \nu(0)e^{-\gamma t}$ and a variance of $Var_0[\nu(t)] = \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma t})$.

---

[17] Bergstrom (1990, p. 1) notes that for large corporations, financial aggregates such as the book value of equity, sales, and expenses are typically the outcome of a huge number "of decisions taken by different individuals at different points in time. For most such variables there will be thousands of small changes at random intervals of time on a single day, and changes can occur at any time during the day. A realistic aggregate model which, accurately, takes account of the microeconomic decision processes must, therefore, be formulated in continuous time." See Bergstrom (1996) and Sims (1971) for further discussion of the advantages and disadvantages of continuous time as against discrete time models, especially in relation to the temporal aggregation and identification issues that arise with these modelling procedures.

Now without loss of generality consider a sample of n firms with identically distributed but orthogonal information variables. [18] It then follows that the expected average information variable, $\overline{\nu(t)} = \frac{1}{n}\sum_{j=1}^{n} \nu_j(t)$, across these n firms will be:

$$E_0[\overline{\nu(t)}] = \overline{\nu(0)}e^{-\gamma t} \qquad \text{(A4)}$$

Likewise, the variance of the average information variable, $Var_0[\overline{\nu(t)}]$, will be:

$$Var_0[\overline{\nu(t)}] = \frac{\sigma^2}{2\gamma n}(1 - e^{-2\gamma t}) \qquad \text{(A5)}$$

Note how this result implies that $\underset{n \to \infty}{\text{Limit}} \; Var_0[\overline{\nu(t)}] \to 0$. This in turn means that $\underset{n \to \infty}{\text{Limit}} \; \overline{\nu(t)} = \overline{\nu(0)}e^{-\gamma t}$ almost surely, in which case under the Ohlson (1995) model one has:

$$\underset{n \to \infty}{\text{Limit}} \; \frac{\overline{\eta(t)}}{\overline{b(t)}} = c_1 + c_2\frac{\overline{x(t)}}{\overline{b(t)}} + c_3\frac{\overline{\nu(0)}}{\overline{b(t)}}.e^{-\gamma t} \qquad \text{(A6)}$$

where $\overline{\eta(t)} = \frac{1}{n}\sum_{j=1}^{n} \eta_j(t)$ is the average equity value, $\overline{b(t)} = \frac{1}{n}\sum_{j=1}^{n} b_j(t)$ is the average book value of equity and $\overline{x(t)} = \frac{1}{n}\sum_{j=1}^{n} x_j(t)$ is the average earnings, all at time t. This means that in "steady state" ($t \to \infty$) and for "large" n, average equity value divided by average book value will bear the following approximate relationship to average earnings divided by average book value:

$$\frac{\overline{\eta(t)}}{\overline{b(t)}} \approx c_1 + c_2\frac{\overline{x(t)}}{\overline{b(t)}} \qquad \text{(A7)}$$

---

[18] The results reported here go through with minimal amendments in the case of correlated but not identically distributed normal variates.

Hence under the aggregation procedure employed in the empirical work summarised in Section 3 of the text the influence of the information variable, $\nu(t)$, is "averaged out" or that to a good approximation, average recursion value is a linear function of average book value and average earnings alone. Hence, if equity prices are based on recursion value alone (so that real options have no role to play in the equity valuation process) then one would expect to see a purely linear relationship between equity prices and earnings using the aggregation procedure employed in this part of our empirical analysis.

**Polynomial Approximation to the Equity Valuation Function**

The first two Laguerre polynomials are $L_0(h) = 1$ and $L_1(h) = 1 - h$ and these polynomials are orthonormal under the inner product $\langle f;g\rangle = \int_0^\infty f(h)g(h)e^{-h}dh$ (Carnahan, Luther and Wilkes, 1969, p. 100). Given this, consider the line of "best" fit to the equity valuation function in terms of these first two Laguerre polynomials:

$$P(h,1,B\theta,\tfrac{1}{2}) = h + \frac{1}{2}\int_{-1}^{1}\exp\left(\frac{-2B\theta h}{1+z}\right)dz \approx \alpha_0 + \alpha_1(1 - h) \qquad \text{(A8)}$$

where $\alpha_0$ and $\alpha_1$ are known as the "Fourier coefficients" of the equity valuation equation, $P(h,1,B\theta,\tfrac{1}{2})$, with respect to $L_0(h)$ and $L_1(h)$, respectively. Now standard results show [Apostol (1969, pp. 29-30)]:

$$\alpha_0 = \langle L_0(h);P(h,1,B\theta,\tfrac{1}{2})\rangle = \langle 1;P(h,1,B\theta,\tfrac{1}{2})\rangle = \int_0^\infty e^{-h}\left[h + \frac{1}{2}\int_{-1}^{1}\exp\left(\frac{-2B\theta h}{1+z}\right)dz\right]dh \qquad \text{(A9)}$$

and:

$$\alpha_1 = \langle L_1(h);P(h,1,B\theta,\tfrac{1}{2})\rangle = \langle 1 - h;P(h,1,B\theta,\tfrac{1}{2})\rangle = \int_0^\infty e^{-h}(1 - h)[h + \frac{1}{2}\int_{-1}^{1}\exp(\frac{-2B\theta h}{1 + z})dz]dh \quad \text{(A10)}$$

Note, however, that the expression for $\alpha_0$ may be decomposed into two integrals, the first of which is $\int_0^\infty he^{-h}dh = 1$. For the second component, note that all functions under the integral sign are continuous in which case we have:

$$\frac{1}{2}\int_0^\infty\int_{-1}^{1} e^{-h}\exp(\frac{-2B\theta h}{1 + z})dzdh = \frac{1}{2}\int_{-1}^{1}\int_0^\infty \exp(\frac{-(2B\theta h + (1 + z)h)}{1 + z})dhdz \quad \text{(A11)}$$

where Fubini's Theorem [Apostol (1969, p. 363)] allows the order of integration to be reversed. One can then evaluate this double integral as follows:

$$\frac{1}{2}\int_{-1}^{1}\int_0^\infty \exp(\frac{-(2B\theta h + (1 + z)h)}{1 + z})dhdz = \frac{1}{2}\int_{-1}^{1}\frac{(1 + z)}{(2B\theta + (1 + z))}dz = \theta B\log(\frac{\theta B}{1 + \theta B}) + 1 \quad \text{(A12)}$$

It then follows:

$$\alpha_0 = \langle L_0(h);P(h,1,B\theta,\tfrac{1}{2})\rangle = \langle 1;P(h,1,B\theta,\tfrac{1}{2})\rangle = \quad \text{(A13)}$$

$$\int_0^\infty he^{-h}dh + \frac{1}{2}\int_{-1}^{1}\int_0^\infty \exp(\frac{-(2B\theta h + (1 + z)h)}{1 + z})dhdz = \theta B\log(\frac{\theta B}{1 + \theta B}) + 2$$

Similar, though more complicated calculations show that the Fourier coefficient with respect to $L_1(h)$ is:

$$\alpha_1 = \langle L_1(h);P(h,1,B\theta,\tfrac{1}{2})\rangle = \langle 1 - h;P(h,1,B\theta,\tfrac{1}{2})\rangle = -1 - \frac{B\theta}{1 + B\theta} - \theta B\log(\frac{\theta B}{1 + \theta B}) \tag{A14}$$

Given these results, it follows that the best linear approximation to the equity valuation equation will be:

$$P(h,1,B\theta,\tfrac{1}{2}) \approx \alpha_0 + \alpha_1(1 - h) = [\theta B\log(\frac{\theta B}{1 + \theta B}) + 2] + [-1 - \frac{B\theta}{1 + B\theta} - \theta B\log(\frac{\theta B}{1 + \theta B})](1 - h)$$

or that:

$$P(h,1,B\theta,\tfrac{1}{2}) \approx \frac{1}{1 + \theta B} + [1 + \frac{B\theta}{1 + B\theta} + \theta B\log(\frac{\theta B}{1 + \theta B})].h \tag{A15}$$

is the best linear approximation to the valuation function $P(h,1,B\theta,\tfrac{1}{2})$ over the semi-infinite real line.

Non-linear approximations to the equity valuation function can be obtained using the higher order Laguerre polynomials based on the recursion formula (Carnahan, Luther and Wilkes, 1969, p. 100):

$$nL_n(h) = (2n - 1 - h)L_{n-1}(h) - (n - 1)L_{n-2}(h) \tag{A16}$$

where $L_n(h)$ is the Laguerre polynomial of order $n \geq 2$. Moreover, one can use this expression to show that the Fourier coefficient, $\alpha_n$, for the equity valuation function with respect to the nth degree Laguerre polynomial, $L_n(h)$, will be:

$$\alpha_n = \frac{B\theta(1 + B\theta)^n - (B\theta)^n(n + B\theta)}{n(n - 1)(1 + B\theta)^n} \tag{A17}$$

again provided $n \geq 2$. It then follows that the Ashton, Cooke and Tippett (2003) equity valuation function can be expressed in terms of the following infinite series expansion:

$$P(h,1,B\theta,\tfrac{1}{2}) = \sum_{n=0}^{\infty} \alpha_n L_n(h) \qquad \text{(A18)}$$

One can illustrate these latter results by letting n = 2 in the recursion formula for the Laguerre polynomials in which case one has that the Laguerre polynomial of order two will be $2L_2(h) = (3 - h)L_1(h) - L_0(h)$ or $L_2(h) = \frac{1}{2}(h^2 - 4h + 2)$. The Fourier coefficient, $\alpha_2$, for the equity valuation function with respect to $L_2(h)$ will then be:

$$\alpha_2 = \langle L_2(h);P(h,1,B\theta,\tfrac{1}{2})\rangle = \frac{B\theta(1 + B\theta)^2 - (B\theta)^2(2 + B\theta)}{2.(2 - 1)(1 + B\theta)^2} = \frac{1}{2}.\frac{B\theta}{(1 + B\theta)^2} \qquad \text{(A19)}$$

It then follows that the "best" quadratic approximation to the valuation function $P(h,1,B\theta,\frac{1}{2})$ over the semi-infinite real line will be:

$$P(h,1,B\theta,\tfrac{1}{2}) \approx \sum_{n=0}^{2} \alpha_n L_n(h) = \alpha_0 + \alpha_1(1 - h) + \frac{\alpha_2}{2}(h^2 - 4h + 2)$$

or, upon collecting terms:

$$P(h,1,B\theta,\tfrac{1}{2}) \approx \frac{\frac{3}{2}B\theta + 1}{(1 + B\theta)^2} + [1 + \frac{(B\theta)^2}{(1 + B\theta)^2} + \theta B\log(\frac{\theta B}{1 + \theta B})].h + \frac{\frac{1}{4}B\theta}{(1 + B\theta)^2}.h^2 \qquad \text{(A20)}$$

Moreover this quadratic approximation to the valuation function $P(h,1,B\theta,\frac{1}{2})$ will be an improvement on the linear approximation summarised earlier. This follows from the fact that the squared error from approximating $P(h,1,B\theta,\frac{1}{2})$ as a linear sum of the first m Laguerre polynomials is given by the squared norm: [19]

$$\left\| P(h,1,B\theta,\tfrac{1}{2}) - \sum_{n=0}^{m} \alpha_n L_n(h) \right\|^2 = \left\| P(h,1,B\theta,\tfrac{1}{2}) \right\|^2 - \sum_{n=0}^{m} \alpha_n^2 \qquad \text{(A21)}$$

[19] A simple proof of this result is given on the Wolfram Mathworld web-site: http://mathworld.wolfram.com/BesselsInequality.html

where $\| P(h,1,B\theta,\frac{1}{2}) \|^2 = \langle P(h,1,B\theta,\frac{1}{2});P(h,1,B\theta,\frac{1}{2})\rangle = \int_0^\infty e^{-h}[h + \frac{1}{2}\int_{-1}^{1}\exp(\frac{-2B\theta h}{1+z})dz]^2 dh$ is

the squared norm of the valuation function and $\alpha_n = \langle L_n(h);P(h,1,B\theta,\frac{1}{2})\rangle$ is the Fourier coefficient of the equity valuation function with respect to the nth order Laguerre polynomial. Since $\alpha_n^2 \geq 0$ for all n it follows from the right hand side of equation (A21) that the squared error, $\| P(h,1,B\theta,\frac{1}{2}) - \sum_{n=0}^{m} \alpha_n L_n(h) \|^2$, declines as the order of polynomial approximation is increased. Indeed, letting $m \to \infty$ in this expression for the squared error leads to Parseval's relation, namely (Clark, 2000, p. 194):

$$\| P(h,1,B\theta,\tfrac{1}{2}) \|^2 = \sum_{n=0}^{\infty} \alpha_n^2 \qquad \text{(A22)}$$

Now, from equation (A17) when $n \geq 2$ we have:

$$\alpha_n^2 = \frac{B^2\theta^2}{n^2(n-1)^2} - \frac{2B\theta}{n}.\{\frac{1}{1+(B\theta)^{-1}}\}^n.\frac{(1+\frac{B\theta}{n})}{(n-1)^2} + \{\frac{1}{1+(B\theta)^{-1}}\}^{2n}.\frac{(1+\frac{B\theta}{n})^2}{(n-1)^2} \qquad \text{(A23)}$$

Consider the last term on the right hand side of this expression, namely:

$$\{\frac{1}{1+(B\theta)^{-1}}\}^{2n}.\frac{(1+\frac{B\theta}{n})^2}{(n-1)^2} \leq \frac{(1+B\theta)^2}{(n-1)^2} \qquad \text{(A24)}$$

This in turn means by the Weierstrass M test that the series expansion $\sum_{n=2}^{\infty} \{\frac{1}{1+(B\theta)^{-1}}\}^{2n}.\frac{(1+\frac{B\theta}{n})^2}{(n-1)^2}$ is uniformly and absolutely convergent over the semi-infinite real line (Spiegel, 1974, p. 228). Similar considerations show that all other terms in the series expansion for $\sum_{n=2}^{\infty} \alpha_n^2$ are uniformly and absolutely convergent over the semi-infinite real line. Given this, one can define the uniformly and absolutely convergent pseudo R-squared statistic:

$$0 \leq R_m^2 = \frac{\sum_{n=0}^{m} \alpha_n^2}{\| P(h,1,B\theta,\frac{1}{2}) \|^2} \leq 1 \qquad \text{(A25)}$$

which gives the proportion of the squared variation in the equity valuation function which is associated with the best fitting mth order linear combination of Laguerre polynomials. It is then possible to use $\alpha_0 = \theta B\log(\frac{\theta B}{1 + \theta B}) + 2$, $\alpha_1 = -1 - \frac{B\theta}{1 + B\theta} - \theta B\log(\frac{\theta B}{1 + \theta B})$ and for $n \geq 2$, $\alpha_n = \frac{B\theta(1 + B\theta)^n - (B\theta)^n(n + B\theta)}{n(n - 1)(1 + B\theta)^n}$ to evaluate the expression for $R_m^2$ and thereby "spectrally decompose" the equity valuation function into its various linear and non-linear components.

## <u>Evaluating the Real Option Integral in the Equity Valuation Function</u>

Under the branching model ($\delta = \frac{1}{2}$) from equation (7) of the text the market value of equity (per unit of book value) is calculated in terms of the following formula:

$$P(h,1,\theta B,\tfrac{1}{2}) = h + \frac{1}{2}\int_{-1}^{1} \exp(\frac{-2\theta Bh}{1 + z})dz = \frac{P(\eta,B,\theta,\frac{1}{2})}{B}$$

where $\eta$ is the recursion value of equity, B is the book value of equity on some given fixed date, $h = \frac{\eta}{B}$ is the normalised recursion value of equity and $\theta$ is the risk parameter that captures the relative stability with which the recursion value of equity evolves over time. Now suppose the firm's normalised recursion value is $h = \frac{\eta}{w} = \frac{\eta}{B} = 1.25$ whilst $\theta B = 2$. Then one can evaluate the integral term in the above formula using the statistic $y = \theta Bh$ in conjunction with Table 2.[20] For the particular values given here we have

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TABLE TWO ABOUT HERE

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$y = \theta Bh = 2\times 1.25 = 2.50$ and so from Table 2, it follows that $\frac{1}{2}\int_{-1}^{1}\exp(\frac{-2y}{1+z})dz = 0.0198$.

This in turn means that the ratio of the market value of equity to the book value of equity will be $\frac{P(\eta,B,\theta,\delta)}{B} = 1.25 + 0.0198 = 1.2698$. In other words, most of the firm's equity value is comprised of recursion value, although if the book value of equity is "large enough" the real option component of equity can still be significant in absolute terms.

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[20] The numerical values reported in Table 1 were obtained using 15 point Gauss-Legendre quadrature. This procedure integrates polynomials of degree 29 or less exactly [Carnahan, Luther and Wilkes (1969, pp. 101-105)].

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