Leverage and Risk Taking

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Abstract

We study a dynamic contracting problem in which size is relevant. The agent may take on excessive risk to enhance short-term gains, which exposes the principal to large, infrequent losses. To preserve incentive compatibility, the optimal contract uses size as an instrument; there is downsizing on the equilibrium path. The contract may be implemented using the full array of financial securities or as a regulation contract with a leverage ratio. We show that holding equity is essential to curb risk taking. Firms that are less prone to risk taking can afford a higher leverage.

Keywords: asymmetric information; dynamic contracts; moral hazard; risk taking.

JEL Classification: D82, D86, G28, L43.

1   Introduction

It is generally accepted that the Global Financial Crisis (henceforth GFC) was precipitated by excessive risk taking on the part of financial institutions that were too highly levered (Duffie (2010); Geithner (2014)). Why exactly did they choose that path remains poorly understood and the complexity of the modern financial system allows only for an imperfect answer.

In this paper we show that leverage and risk taking are intimately connected: the incentives to take on (excessive) risk depend on the leverage of the speculator, for the simple reason that (firm) size governs the returns on risk taking. The implication is then quite simple: in order to control risk taking, one must control leverage. This conclusion provides foundations for equity requirements on firms, especially on financial institutions. It departs from the commonly–accepted wisdom that equity capital is required to sustain adverse shocks, such as economic downturns, or to absorb losses (see, e.g. Bonaccorsi di Patti et al. (2015)), which was challenged by Diamond and Rajan (2000)[1]. Instead, equity requirements, or a limit on financial leverage, are necessary to present the the potential speculator with the appropriate incentives and to curb risk–taking. In other words, they are prevention measures rather than crisis–management tools. In light of the GFC and of the many isolated incidents since (Société Générale, 2008, €4.9 Bn; UBS, 2011, US\$ 2Bn or the London Whale at JP Morgan, 2012, US\$ 7Bn) we believe it is an important contribution to better understand the role of leverage in *fostering* crises – rather than just in amplifying them.[2]

In order to make our point we study a stylized, continuous–time contracting model under moral hazard, in which an agent controls the profitability of a project through a scalable arithmetic Brownian motion. The agent must be provided incentives to not divert funds (equivalently, to exert effort); she can also engage in excessively risky activities that increases the drift in the operating cash–flows process but it exposes the firm to a Poisson process of very large losses. Put differently, she may take on tail risk on the asset side of the firm's balance sheet. Thanks to the aforementioned scale effect, the model exhibits rich dynamics.

Firm size emerges as a necessary control to satisfy incentive compatibility because the returns on risk taking are increasing in the size of the project. Risk taking is deterred

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[1]Diamond and Rajan (2000) point out that equity is not necessary in order to cover credit losses in a bank; any claim subordinated to deposits is sufficient; thus, long–term debt is sufficient to guarantee the depositors' confidence and to solve the issue of bank runs.

[2]An asset bubble that is fueled by leverage (credit) is known to have deeper and longer–lasting effects than other crises (Geithner (2014)) because a levered failure affects the lender as well as the borrower and depresses collateral value: it spreads. We do not study the unraveling of leverage but its effect on incentives to take risks.

when the agent's continuation utility (her stake in the project) is sufficiently large. That is, the agent must have enough to lose so as to not speculate. The incentive compatible, optimal contract includes downsizing along the equilibrium path in order to preserve incentive compatibility. We characterize this downsizing process.

The downsizing option is only relevant at the lower boundary of the domain of the differential equation that characterizes the principal's value function; hence, all the firm–size dynamics are centered at that boundary. Whenever the principal downsizes the project, he decreases that lower boundary as well: it is a floating boundary. If downsizing has to be so severe that the firm becomes inefficiently small, the principal prefers liquidating; we show there exists an optimal liquidation size. A floating boundary is cumbersome and does not easily lend itself to interpretation. Thankfully, the value function is homogeneous of degree one in the size variable, which has two benefits. First, in size–adjusted terms the floating boundary becomes a more standard reflecting barrier. Reaching it triggers downsizing, which, in size–adjusted terms, results in reflection. Second, away from the lower boundary, the differential equation that characterizes the (scale adjusted) value function is the same as that of DeMarzo and Sannikov (2006), the behavior of which is well understood. In particular, it features a reflecting payment barrier $\widetilde{w}$, beyond which the principal pays cash out to the agent. However, here the exact location of the payment boundary depends on severity of the risk–taking problem. The comparative statics indicate that the value to the principal uniformly decreases when risk taking becomes more lucrative.

We suggest two implementations to connect our mathematical results to practical questions in finance. For any implementation, the risk–taking incentives affect both the value of the firm and its debt capacity. Quite intuitively, the milder the risk taking problem, the more valuable is the cash-flow stream, and the more debt a firm can carry to finance itself. That is, in equilibrium less–risky firms are more highly levered: they face a smaller equity requirement. First, the optimal contract can be implemented as a capital structure using standard securities, but it requires a richer set of instruments than the implementation of DeMarzo and Sannikov (2006). In particular, it is essential that the agent holds enough equity to deter her from risk taking – and not just to allocate her cash–flow rights. This

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is achieved by adding covenants to the debt contract. In addition, to cope with downsizing the debt has to be fully flexible. In this interpretation, leverage measures the normalized value of the inside equity; it can easily be converted to an accounting metric by using the book–to–market ratio of the firm. Second, we also implement the contract through banking regulation, which compels the use of mark–to–market accounting and allows for a requirement on the book value of equity with an intervention threshold. This is well in line with current practice.

Finally, we develop two extensions: The first one studies the opportunity to invest in the firm to increase its size. The principal only invests when the agent's continuation value is sufficiently high, so as to avoid premature downsizing following costly investment. In the second extension, we suppose the project cannot be liquidated; for example, a large bank should not stop operating. This fits large financial institutions such as Global Systemically Important Banks (there are 29 worldwide) and Global Systemically Important Insurers (there are 9). The contractual stopping time is then a restructuring threshold at which the agent is terminated, but the firm is no liquidated.

The papers closest to this work are Biais et al. (2010) (henceforth BMRV), He (2009), Biais and Casamatta (1999), as well as DeMarzo et al. (2013) and Rochet and Roger (2015). He (2009) extends the work of DeMarzo and Sannikov (2006) (henceforth DS) to a model where the cash–flow dynamics follow a geometric Brownian motion. Incentive compatibility is independent of scale and size cannot be a control. BMRV study the problem of large Poisson risks (accidents or losses), the probability of which is controlled through the agent's effort; there is no Brownian component. Incentive compatibility dictates that the agent's continuation value be decreased following a loss; it occurs along the equilibrium path because the probability of a loss is never zero. Sometimes restoring incentive compatibility requires the firm to be downsized too. Here movements in the agent's continuation value are driven by the Brownian process, not by episodic (Poisson) jumps. We also find that downsizing is necessary and occurs in equilibrium, however it is used to deter risk taking, not to spur effort. Last, our downsizing is continuous and is driven by the Brownian process, not by the Poisson one.

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Biais and Casamatta (1999), like Rochet and Roger (2015) and DeMarzo et al. (2013) (independently) study the double moral–hazard problem of diversion (or effort) and risk taking. Biais and Casamatta (1999) use a static model; their optimal contract always includes debt and equity, and sometimes options. Debt turns the agent into the residual claimant and so it spurs effort, as established by Innes (1990). Options may be required to complement that effect. In our dynamic model, movements in the continuation value replace options. In Biais and Casamatta (1999) equity is necessary to deter risk shifting; this is analogous to our use of equity to deter excessive risk taking. Rochet and Roger (2015) and DeMarzo et al. (2013) develop continuous–time versions using arithmetic Brownian motions. Incentive compatibility requires the agent's continuation value to remain sufficiently high, but it is size independent. We augment these works by adding an important size effect that is easily interpreted as leverage. Further, downsizing, instead of termination in those models, can be used to preserve incentive compatibility.

This paper also connects to a large literature on leverage and risk–taking, especially in financial institutions. In a series of papers, Anat Admati (and at times her coauthors) makes the argument for less leverage on the grounds of less fragility for individual firms, less subsidies from society and greater systemic resilience (see, e.g. Admati (2014)). In that perspective, leverage amplifies the transmission of shocks and the severity of crises. This paper adds a simple but salient point: with less levered institutions, there is also less risk–taking, which reinforces financial stability. VanHoose (2007) provides a survey that suggests a persistent lack of consensus on the role and benefit of capital requirements. Furlong and Keeley (1989) establish that asset risk decreases when the capitalization of a bank increases. Milne (2002) observes that a bank's portfolio choice depends on its capitalization. This model accords well with both, and minimum capital requirements induce the institution to choose the less risky path. The reason is that breaching the capital requirement triggers downsizing, which is costly. Morrison and White (2005) propose a model of adverse selection and moral hazard in which capital requirements are also used to solve the moral–hazard problem and to screen out bad banks (or bankers). Last, this work connects to a more recent literature on interventions and bailouts. Here the resolution mechanism is preemptive (to

preserve incentive compatibility); it anticipates a crisis, which requires strong and attentive supervision.

2 Model

We consider a principal–agent problem set in a continuous–time framework. The principal (shareholders or a regulator) must rely on the expertise of an agent (manager or firm) to operate their business. All parties are risk–neutral; the principal discounts future payments at rate $r > 0$ and the agent is (weakly) more impatient, as her discount rate $\rho \geq r$. In order to formally describe the principal–agent interaction, let us introduce the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. At each instant $t \geq 0$, the agent chooses an *action* $a_t \in \{0, 1\}$. If the agent chooses the action 0 we say that she has *speculated*, whereas if her choice is 1 we say she has been *prudent*. Let $\boldsymbol{\mu}, \Delta\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ be strictly positive and define the functions

$$\mu(a) := \begin{cases} \boldsymbol{\mu} + \Delta\boldsymbol{\mu}, \text{ if } a = 0; \\ \boldsymbol{\mu}, \qquad \text{ if } a = 1, \end{cases} \quad \text{and} \quad \lambda(a) := \begin{cases} \boldsymbol{\lambda}, \text{ if } a = 0; \\ 0, \text{ if } a = 1. \end{cases} \tag{2.1}$$

For a given strategy $a = (a_t, 0 \leq t)$ we define $N(a) := (N_t(a_t), 0 \leq t)$ as the Poisson process with intensity $\lambda(a) = (\lambda(a_t), 0 \leq t)$. We shall also make use of the standard Brownian motion $Z = (Z_t, 0 < t)$ and, from this point on, we assume that $\mathbb{F}^a := (\mathcal{F}_t^a, t \geq 0) = \sigma(Z, N(a))$ is the natural filtration induced by the agent's strategic choice. In the sequel we denote by $\mathcal{A}$ the set of all admissible strategies for the agent. We are now in a position to introduce the firm's *operating cash flows*, which, for $a \in \mathcal{A}$, follow the process

$$dS_t^a = X_t\big(\mu(a_t)dt + \sigma dZ_t - LdN_t(a_t)\big), \quad S_0 = 0, \tag{2.2}$$

where, for $t \geq 0$, X_t is the firm's size (for a bank one could think of its balance sheet). Indeed, risk taking, or asset shifting, increases the drift of the cash–flows process but introduces exposure to large losses. We assume these losses are sufficient to wipe out the firm, and so are never desired by the principal. We discuss whether the principal ever has incentives to induce risk taking in Section 7.

The $\mathbb{F}^a$–predictable process $X = (X_t, 0 \leq t)$, which is chosen by the principal, is non–increasing. In other words, we allow for downsizing but not for growth; we relax this assumption in Section 6. Downsizing is undesirable for the principal, as it reduces the cash flows, but it may be necessary for incentive purposes. Downsizing implies partial liquidation and for simplicity we suppose it generates no proceeds. The firm's initial size is denoted by X_0 and, for the time being, we take as given some $X_0 \geq \overline{X} > 0$ that corresponds to the minimal project scale that the principal is willing to keep in operation. At the first date $t > 0$ such that $X_t = \overline{X}$, the agent is fired and the firm is liquidated (or restructured). Liquidation earns the principal Π. We first focus on liquidation as it is more tractable, and analyze restructuring as an extension(Section 6).[3] As we show in Theorem 1 below, in equilibrium the choice of $\overline{X}$ depends, among other things, on the magnitude of Π. Intuitively, the principal is not willing to liquidate too–large a firm and receive too–small a compensation for it. We suppose the agent's outside option is zero.

There are two sources of frictions. First, in the spirit of DS, the agent might divert cash flow: she may hand over $d\widehat{S}_t < dS_t^a$ and appropriate the difference. A dollar diverted brings the agent $\eta \leq 1$ dollars, i.e. misreporting results in the instantaneous profits $\eta|dS_t^a - d\widehat{S}_t|$. Second, the agent can secretly engage in excessively risky ("speculative") activities that generate the additional cash–flow $X_t \Delta\mu dt$ per unit of time but expose the firm to the catastrophic losses $X_t L$. For example, the firm sells (but does not buy) CDS (like AIG or Morgan Stanley during the GFC) or issues options. It could also speculate on electricity contracts (like Enron in the late 1990's). Since losses corresponding to a realization of the Poisson risk are catastrophic, there is no loss of generality in assuming that $\widehat{S}$ is a continuous process, which is also $\mathbb{F}^a$–adapted.[4]

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[3] "Restructuring" means the firm cannot be liquidated – for example, a very large bank. Instead the agent is fired and replaced, and a new contract is initialized.

[4] We rule out private savings by the agent without loss of generality (see DS for details). Hence, we do not require $S - \widehat{S}$ to be of bounded variation. The difference here is that, since the agent is devoid of the possibility to over–report, we do not split the instantaneous effects of diversion/over–reporting into their positive and negative components (the term $[Y - \widehat{Y}]^\lambda$ in DS). This is only well defined under the bounded–variation hypothesis, but does not play a role here.

3 The Contract

The principal seeks to maximizes the ex–ante value of the firm in the presence of moral hazard. Given our assumption of very large losses L, it is optimal for the principal to always deter risk taking. The contract between the principal and the agent is designed and agreed upon at date $t = 0$ and we assume that all parties can commit to it. A contract $\Xi = (X, I, \tau)$ stipulates, contingent on the history of observed cash flows, a firm–scale process X (or equivalently, a downsizing process $X^d := X_0 - X$), a non–decreasing process I of cumulative payments to the agent and a (random) termination time τ. The fact that I is non–decreasing reflects the agents *limited liability*. For a given contract Ξ, the agent chooses her strategy by solving

$$\sup_{a \in \mathcal{A}} U_0^a(\Xi) := \sup_{a \in \mathcal{A}} \mathbb{E}\left[\int_0^\tau e^{-\rho s}\big(dI_s + \eta|dS_s^a - d\widehat{S}_s|\big)\right] \tag{3.1}$$

The contract is incentive compatible if it is designed in such a way that the agent never finds it optimal to divert cash, nor to engage in speculative activities. The principal takes this problem as a constraint and designs a contract Ξ so as to maximize his net payoff.

Following Spear and Srivastava (1987), who introduced the recursive approach to contracting, any contract can be characterized by the stochastic process W describing the continuation payoff of the agent when the contract Ξ is executed. If the agent chooses strategy a then

$$W_t^a(\Xi) = \mathbb{E}\left[\int_t^\tau e^{-\rho(s-t)}d\widehat{C}_s^a + e^{-\rho(\tau-t)}W_\tau^a \middle| \mathcal{F}_t^a\right], \tag{3.2}$$

where $d\widehat{C}_s^a := dI_s + \eta|dS_s^a - d\widehat{S}_s|$ is the consumption process of the agent. In line with Sannikov (2008), we want to show $W^a(\Xi)$ can be represented as a diffusion process in order to characterize its dynamics. To this end we make use of the Martingale Representation Theorem for jump–diffusion processes (henceforth MRT), which in a nutshell states that any process Y that is a martingale with respect to the filtration $\mathbb{A}$ generated by a Wiener process $B = (B_t, 0 < t)$ and a Poisson process $M = (M_t, 0 < t)$ with intensity γ can be written as

$$Y_t = y_0 - \int_0^t h_s\big[dM_s - \gamma_s ds\big] + \int_0^t g_s dB_s,$$

where $h = (h_t, t \geq 0)$ and $g = (g_t, t \geq 0)$ are unique, $\mathbb{A}$–adapted processes in $\mathcal{L}^*$.[5] A thorough exposition of the MRT at hand can be found, for instance, in Applebaum (2009).

Given a contract Ξ and an action $a \in \mathcal{A}$, the we define the agent's total utility at date t as

$$\widehat{\psi}_t^a(\Xi) := \mathbb{E}\left[\int_0^\tau e^{-\rho s} d\widehat{C}_s^a \bigg| \mathcal{F}_t^a\right],$$

which is clearly a $\mathbb{F}^a$–martingale. From this point on we refrain from writing Ξ as an argument, unless there are grounds for confusion. Applying the MRT, there exist $\mathbb{F}^a$–adapted processes $P^a = (P_t^a, t \geq 0)$ and $\beta^a = (\beta_t^a, t \geq 0)$ such $\widehat{\psi}_t^a$ may be written as

$$\widehat{\psi}_t^a = \widehat{\psi}_0^a - \int_0^t e^{-\rho s} P_s^a [dN_s - \lambda(a_s)ds] + \int_0^t e^{-\rho s} \beta_s^a dZ_s, \tag{3.3}$$

where $e^{-\rho s} > 0$ is a scaling factor. Moreover, since $\int_0^t e^{-\rho s} d\widehat{C}_s^a$ is $\mathcal{F}_t^a$–measurable, U_t^a can be rewritten as

$$U_t^a = \int_0^t e^{-\rho s} d\widehat{C}_s^a + e^{-\rho t} W_t^a. \tag{3.4}$$

Given that $\widehat{\psi}_t^a$ is a martingale, Equations (3.3) and (3.4) imply

$$\widehat{\psi}_0^a - \int_0^t e^{-\rho s} P_s^a [dN_s - \lambda(a_s)ds] + \int_0^t e^{-\rho s} \beta_s^a dZ_s = e^{-\rho t} W_t^a + \int_0^t e^{-\rho s} d\widehat{C}_s^a. \tag{3.5}$$

In differential form, after rearranging terms and eliminating the factor $e^{-\rho t}$, Expression (3.5) becomes

$$dW_t^a = \rho W_t^a dt + \beta_t^a dZ_t - d\widehat{C}_t^a - P_t^a [dN_t - \lambda(a_t)dt], \tag{3.6}$$

which is a jump–diffusion process. Here β_t^a and P_t^a represent the sensitivity of the agent's continuation payoffs to the volatility of cash flows and to large losses, respectively. As in DS and others since, incentive compatibility can be characterized by simple conditions on these sensitivity parameters. However, because we must account for both diversion and risk taking, our results depart from DS, Sannikov (2008) and He (2009), for instance.

Proposition 1 *For any strategy $a \in \mathcal{A}$ there is no cash diversion as long as*

$$\beta_t^a \geq \eta\sigma =: \beta, \tag{3.7}$$

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[5]Processes in the $\mathcal{L}^*$ space are square-integrable with finite expected value: $\mathbb{E}\left[\int_0^t h_s^2 ds\right] < \infty$.

and there is no risk taking, i.e. $a_t \equiv 1$, if and only if

$$P_t^a = P_t \geq \eta \frac{\Delta\mu}{\lambda} X_t \tag{3.8}$$

Combining Expressions (3.7) and (3.8), we have that a contract deters both risk taking and cash diversion only if

$$P_t \geq \frac{\beta_t}{\sigma} \frac{\Delta\mu}{\lambda} X_t,$$

and by limited liability[6]

$$W_t \geq P_t \geq \frac{\beta_t}{\sigma} \frac{\Delta\mu}{\lambda} X_t =: w_m X_t =: W_m(X_t). \tag{3.9}$$

The first part of Proposition 1 is akin to the results of DS and He (2009). By exposing the agent to the random part of the cash–flow, the principal can deter diversion because the agent steals from her own pocket. Given that for any $a \in \mathcal{A}$ both the drift and the stochastic drivers of dS_t^a are scaled by X_t, said exposure is independent of the firm's current size. Therefore, the latter does not enter Constraint (3.7). To gain some intuition, observe that the term

$$\left[\frac{\beta_t^a}{\sigma} - \eta \right] \left[dS_t^a - d\widehat{S}_t^a \right]$$

showcases the compromise the agent makes when deciding whether to divert funds: she enjoys the instantaneous benefit $\eta[dS_t^a - d\widehat{S}_t^a]$ from immediate consumption but forgoes $[\beta_t^a/\sigma][dS_t^a - d\widehat{S}_t^a]$ – the decrease in her continuation value. This is true for any strategy a, as we show formally in the Appendix. Due to the fact that the current scale appears on both sides of this compromise it is neutral on the agent's incentives to divert funds.

Turning to risk taking, consider two strategies as follows: $a_t \equiv 1$ (always prudent) and $\tilde{a} = \{a_s = 0, s < t; a_s = 1, s \geq t\}$ (take excessive risk until date t, then be prudent), for some $t > 0$. Since s, t are arbitrary and Bellman's Principle of Optimality applies, what follows holds true for any strategy a. To deter risk taking, for any processes C_t and β_t^a, the principal needs to make the expected penalty

$$\int_0^t e^{-\rho s} \lambda P_s^{\tilde{a}} ds$$

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[6]The reason for relabeling $\beta_t \Delta\mu/(\sigma\lambda)$ as w_m will become apparent in Section 4.2.

larger than the corresponding expected gains

$$\int_0^t e^{-\rho s} \left(d\widetilde{C}_s^{\tilde{a}} - d\widetilde{C}_s^1 \right) ds,$$

representing an increase in consumption generated by following strategy $\tilde{a}$ until time t. This must hold regardless of whether $d\widetilde{C}_s = d\widehat{C}_s$ (diversion) or $d\widetilde{C}_s = dC_s$ (not). When there is diversion, we therefore need $P_t \geq \eta \frac{\Delta \mu}{\lambda} X_t$ and when diversion is deterred too, we require $P_t \geq W_m(X_t)$. Unlike the diversion problem, X_t is only present in the right–hand sides of the inequalities; therefore, size does matter for risk taking. This result is novel: Engaging in risk taking generates an additional gain $\Delta \mu X_t$, of which the agent always appropriates a fraction η or β_t/σ. The incentives are strongest when $P_t \equiv W_t$: the agent must be wiped out after a large adverse event. Any further penalty would violate limited liability, thus $W_t \geq W_m(X_t)$ so as to preserve incentive compatibility.

Remark 1 *Condition (3.9) is noteworthy: it requires the continuation value to grow linearly with the size of the firm X_t. This condition arises naturally here and may contribute to explain the increase in executive compensation at large firms (see Gabaix and Landier (2008)).*

4   The value function and the optimal contract

The principal maximizes the discounted, expected cash flows net of payments to the agent over all incentive–compatible contracts, which we denote by IC. In other words, the principal's *value function* is given by

$$V(X, W) := \sup_{\Xi \in IC} \mathbb{E} \left[\int_0^\tau e^{-rs} (dS_s^a - dI_s) \Big| X_0 = X, W_0 = W \right]. \tag{4.1}$$

We conjecture (and later verify) that the optimal contract is such that

$$\beta_t^a = \beta \text{ and } P_t = W_m(X_t).$$

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Given that the principal never wants to allow risk taking, along the optimal path W_t evolves according to the dynamics

$$dW_t = \rho W_t dt - dI_t + \beta X_t dZ_t. \tag{4.2}$$

Under an incentive compatible contract (which determines the dynamics of W), the value function satisfies the following Hamilton–Jacobi–Bellman equation:

$$rV(X_t, W_t)dt = \mu X_t dt + \sup_{dX_t, dI_t} \left\{ -dI_t + (\rho W_t dt - dI_t)V_W(X_t, W_t) + V_X(X_t, W_t)dX_t \right.$$
$$\left. + \frac{\beta^2 X_t^2}{2}V_{WW}(X_t, W_t)dt \right\}, \tag{4.3}$$

subject to the incentive compatibility Constraints (3.7) and (3.9). Since X is a decreasing process it is of bounded variation. Moreover, as we show below, X has continuous paths, therefore the cross–variation term $\langle X, W \rangle_t$ and the quadratic variation $\langle X, X \rangle_t$ are both zero. As a consequence there are no V_{XW} nor V_{XX} terms in Equation (4.3). The firm's scale X appears as a state variable in Equation (4.3), whereas dX_t is a control. This obeys the non–standard structure of our problem, where the firm's current size matters for both incentives and continuation values. X_t enters the equations that determine the dynamics of V and W, but its impact does not stop there. Indeed (i) the agent is subject to downsizing at any point in time to preserve incentive compatibility – Condition (3.9) – and (ii) if downsizing leads the firm to become too small, it may be terminated:

$$V(\overline{X}, W_m(\overline{X})) = \Pi. \tag{4.4}$$

These conditions define an intervention threshold at $W_m(X_t)$. Equation (4.3) has a free upper boundary that may be determined through an optimality condition. Let $\widetilde{W}(X)$ be characterized by the condition $V_W(X, \widetilde{W}(X_t)) = -1$. As is now commonly known, $\widetilde{W}(X)$ is a *payment barrier*: transfers to the agent are postponed until the time when increasing her continuation utility becomes too expensive for the principal.

We postpone a more precise analysis of $\widetilde{W}(X)$ and instead stress that payments to the agent only take place at $\widetilde{W}(X)$ when $W_m(X) < \widetilde{W}(X)$, i.e. when the reflecting barrier is not in conflict with the incentive constraint (3.9). Whether this applies here depends on the

risk–taking problem too. The complementary case described by $W_m(X) \geq \widetilde{W}(X)$ pits the reflecting barrier $\widetilde{W}(X)$ with the no risk taking constraint (3.9). Then, of course, there can be no payment at $\widetilde{W}(X)$, the contract is not even incentive compatible at that point. In fact the principal must even reduce X_t to restore incentive compatibility.[7] Then (i) $\widetilde{W}$ no longer identifies an upper (payment) boundary and (ii) the upper boundary of $V(X, W)$ and the lower boundary induced by the termination condition (4.4) may be confounded.

Let us assume that $W_m(X) < \widetilde{W}(X)$; then, as long as $W_t \in \big(W_m(X_t), \widetilde{W}(X_t)\big)$, there are neither downsizing nor monetary transfers to the agent. In other words, both dI_t and dX_t are zero on that open interval; then the continuation values for the principal and the agent are characterized by the equations

$$rV(X, W) = \mu X + \rho W V_W(X, W) + \frac{\beta^2 X^2}{2} V_{WW}(X, W) \tag{4.5}$$

and

$$dW_t = \rho W_t dt + \beta X_t dZ_t, \tag{4.6}$$

respectively. As long as the risk–taking constraint remains slack, no changes occur to the payment barrier $\widetilde{W}(X_t)$ and the cumulative payments to the agent are such that W_t is *reflected* downwards. As in DS this reflecting barrier induces a boundary condition for the HJB equation (4.3) that is complemented by the super–contact condition $V_{WW}(X, \widetilde{W}(X)) = 0$. Together these pasting conditions yield

$$rV(X, \widetilde{W}) + \rho \widetilde{W} = \mu X, \tag{4.7}$$

where the scale X also figures, and we have

$$dI_t \begin{cases} = 0, & \text{if } W_t < \widetilde{W}(X); \\ > 0, & \text{if } W_t \geq \widetilde{W}(X). \end{cases}$$

Intuitively speaking dI_t "equals" $\max\big\{0, W_t - \widetilde{W}(X_t)\big\}$, i.e. all the value in excess of the payment barrier is immediately paid out to the agent.[8] At the point $\widetilde{W}$, the continuation

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[7] In DS, He (2009) or BMRV, for example, the unique reflecting barrier is $\widetilde{W}$.

[8] In technical terms, the process I_t is the *local time* of W at the level $\widetilde{W}(X_t)$. We refer the reader to Revouz and Yor (1999) for a thorough exposition of Brownian local times.

value W can no longer grow as it, together with $V(W)$, exhaust the cash flow μX. We stress that, as long as no downsizing occurs, X is simply a parameter in Equations (4.5) and (4.6). In the sequel our strategy is to simplify Equation (4.3) through the identification of the maximizing strategies and characterize its solution.

4.1 Incentive compatibility and downsizing

A key feature that distinguishes our model form that of He (2009) is the principal's option to downsize. This is necessary when the risk taking constraint $W_t \geq W_m(X_t)$ is violated, for then the agent has incentives to speculate. However, as long as $X_t > \overline{X}$ termination is not necessary to preserve incentives. Incentive compatibility may be restored: the principal can simply downsize. That is, the incentives Constraint (3.9) induces a *floating* (lower) boundary at $W_m(X_t)$, which is clearly a function of X_t. Intuitively speaking, the floating boundary tracks the downward component of the agent's continuation utility (the negative increments of W_t) once the constraint becomes binding. Furthermore, when this occurs the dynamics of W are also impacted by the firm's downsizing, since the corresponding volatility is linear in X.

Recall that $w_m = \beta \Delta \mu / (\sigma \lambda)$ and define $R_t := \inf \{W_s, 0 \leq s \leq t\}$, the *running infimum* of W up to time t, i.e. the smallest value that the agent's continuation value has attained during her tenure. As long as $R_t > w_m X_0$ no downsizing has been required to preserve incentive compatibility. As soon as that barrier is reached, we have $X_t = R_t/w_m$. Put differently, as long as the incentive constraint remains slack, X_t is constant and $W_t \in \left(W_m(X_t), \widetilde{W}(X_t)\right)$. We also know that, whenever $W_t = \widetilde{W}(X_t)$, $dI_t > 0$. In fact, at this point if W "pushes upwards" it stays pegged at $\widetilde{W}(X_t)$. Loosely speaking, as soon as W stops increasing, it is reflected downwards by its own dynamics; reflection follows the fact that the process I prevents W from exceeding $\widetilde{W}(X_t)$. With downsizing the dynamics are quite similar, and it starts as soon as $W_t = w_m X_t$ if W "pushes downwards". Indeed,

Lemma 1 *Downsizing is necessary as soon as $W_t = W_m(X)$; more precisely, the distribu-*

tions of the first–visitation and the first–crossing times

$$\tau_v = \inf\left\{t \geq 0 | W_t = W_m(X)\right\} \quad and \quad \tau_c = \inf\left\{t \geq 0 | W_t < W_m(X)\right\},$$

respectively, are identical.

As soon as W stops pushing downwards, the incentive constraint becomes slack again and downsizing stops. This is formalized by setting

$$X_t = \min\left\{R_t/w_m, X_0\right\}. \tag{4.8}$$

Then the *cumulative downsizing process* X^d is given by $X_t^d = X_0 - X_t$. Loosely speaking, X^d counts the time that the agent's continuation value spends at the boundary $W_m(X)$, i.e. the length of each downsizing period.

Result: The optimal downsizing policy is given by $X_t^d = X_0 - X_t$, where $X_t = \min\left\{R_t/w_m, X_0\right\}$.

We note that under the assumption that $W_m(X) < \widetilde{W}(X)$, it is clear that there should be no downsizing as long as $dI_t > 0$. Instead of downsizing the principal could simply set $dI_t = 0$, which would increase W_t one–to–one. In Figure 1 we depict the value function corresponding to the scales $X_0 = 1$ and $\overline{X} = 0.4$. and the parameters $\mu = 0.6$. $\beta = 0.8$, $r = 0.25$, $\rho = 0.3$, $w_m = 0.7$, and $\Pi = 0.414$. The scale–dependent payout and downsizing barriers, $\widetilde{W}(X)$ and $W_m(X)$ respectively, are highlighted.

4.2 Homogeneity of $V(X, W)$

In order to study the solution to Equation (4.3) we make use of the fact that, by virtue of the cash–flow process S being linear in X, the value function V is homogeneous in X, as in He (2009) and BMRV. In other words, there exists a function $v : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$V(X, W) = Xv\left(\frac{W}{X}\right). \tag{4.9}$$

If we let $w_t := W_t/X_t$ we may write the size–adjusted version of Equation (4.3) as

$$rv(y)dt = \mu dt + \sup_{di_t, dx_t} \left\{ -di_t + v'(y)(\rho w_t dt - di_t) + (v(y) - yv'(y))dx_t + \frac{\beta^2}{2}v''(y)dt \right\}, \tag{4.10}$$
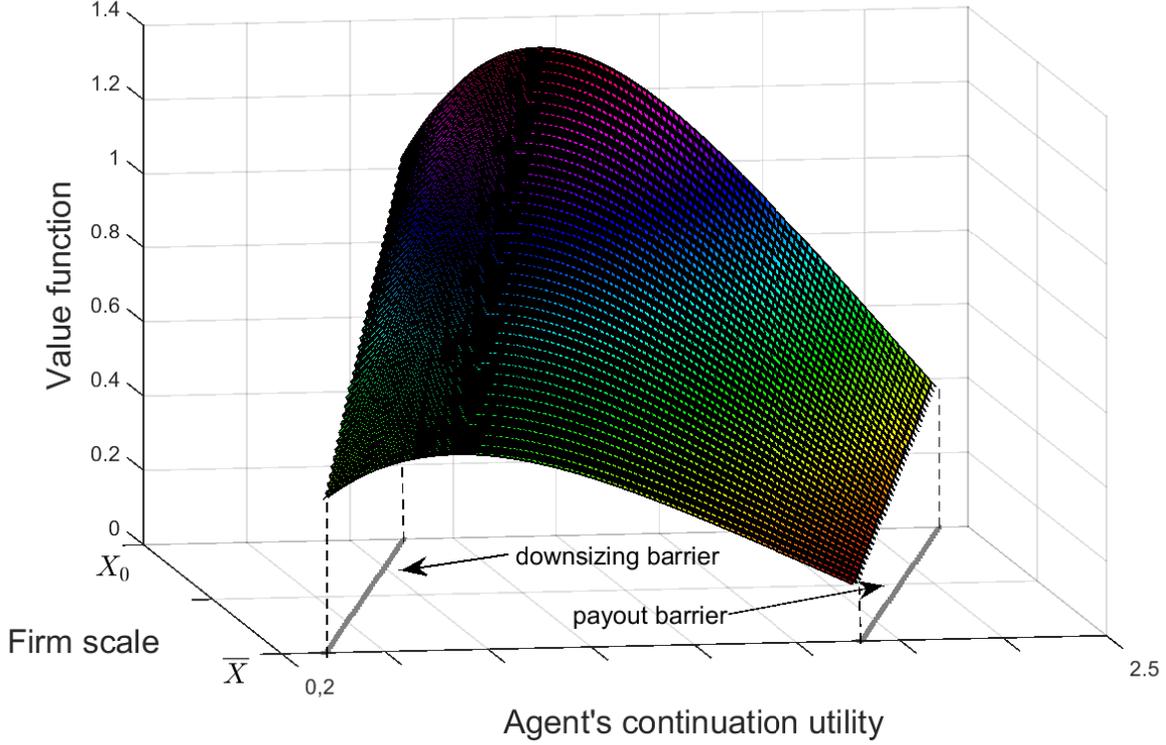
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Figure 1: Value function and the payout and downsizing barriers

where $di_t := dI_t/X_t$, and $dx_t := dX_t/X_t$, subject to the no–diversion Constraint (3.7) as before, together with the size–adjusted, risk taking constraint

$$w_t \geq w_m. \tag{4.11}$$

The scale–adjusted HJB Equation (4.10) resembles that of He (2009) – less the cross–derivatives for the reasons presented previously – and to some extent BMRV. It does not, however, feature a discontinuous downsizing process, which BMRV must use to preserve incentive compatibility after observing losses. Instead, our downsizing process is continuous and, importantly, it is used preemptively with the sole purpose of deterring risk taking.

In size–adjusted terms, the floating barrier $W_m(X_t)$ becomes a reflective one at w_m for the size–adjusted continuation value w_t of the agent. Using Itô's formula

$$dw_t = \frac{dW_t}{X_t} + W_t \left(-\frac{dX_t}{X_t^2} + \frac{1}{X_t^3} d\langle X, X \rangle_t \right) + d\langle W, 1/X \rangle_t$$

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We know that X is continuous and, since it is bounded away from zero, $1/X$ is continuous as well. Therefore, $\langle X, X \rangle_t = \langle W, 1/X \rangle_t = 0$, which yields

$$dw_t = \frac{dW_t}{X_t} - W_t \frac{dX_t}{X_t^2} = \rho w_t dt - di_t + \beta dZ_t - w_t dx_t.$$

Moreover, dx_t only carries mass on the set $\{w_t = w_m\}$, hence

$$dw_t = \rho w_t dt - di_t + \beta dZ_t - w_m dx_t. \tag{4.12}$$

The (size–adjusted) term $-w_m dx_t$ in Equation (4.12) introduces an instantaneous reflection of the process w at the level $w = w_m$. In other words, if we consider the following problem on $[w_m, \tilde{w}]$: find three processes (w, H^*, G^*) that satisfy

$$
\begin{aligned}
& w_t = w_0 + \rho \int_0^t w_s ds + \beta \int_0^t dZ_s - H_t^* - G_t^*, \\
& w_m \leq w_t \leq \tilde{w}, \ t \geq 0, \\
& \int_0^\infty \mathbf{1}_{\{w_t < \tilde{w}\}} dH_t^* = \int_0^\infty \mathbf{1}_{\{w_t > w_m\}} dG_t^* = 0,
\end{aligned}
\tag{4.13}
$$

then for all $t \geq 0$ we have $i_t \equiv H_t^*$ and $w_m x_t \equiv G_t^*$. The solution to the so–called Skorokhod Problem (4.13) is discussed, for instance, in Karatzas and Shreve (1991). Intuitively, as long as $w > w_m$, it holds that $dx_t = 0$. Whenever $w_t = w_m$, the process x is active if w "pushes downwards" (like W in the preceding section) and its role it precisely to maintain the inequality $w_t \geq w_m$. As soon as w "pushes upwards", we have $dx = 0$ and w is reflected upwards by its own dynamics. We prove this formally in

Lemma 2 *The term $-w_m dx_t$ in the dynamics of the process $w = \big(w_t, 0 \leq t\big)$ induces an instantaneous reflection of the latter at the level $w = w_m$.*

The behavior at the boundary $w = w_m$ differs from both DS and Zhu (2013). In DS the lower boundary necessarily induces termination (there is no scale to adjust). In Zhu's model the boundaries are sticky for some classes of contracts. That is, the continuation value w may remain at the lower bound w_m for a positive measure of time because the principal suspends the contract. Then, the stochastic component dZ_t is neutralized and w is allowed

to remain at w until the contract is reactivated. There is no suspension of the process dZ_t here; hence, the reflection is instantaneous (That is not to say downsizing is not costly to the agent; indeed, her scaled continuation value $W = wX$ is clearly lower after downsizing.).

In analogous fashion to the scaled case, there exists $\widetilde{w}$ such that $di_t = 0$ if $w_t < \widetilde{w}$. Hence, the processes i and x are inactive on the open interval $(w_m, \widetilde{w})$. The reflection at level $w = \widetilde{w}$ results in the condition $v'(\widetilde{w}) = -1$, with $\widetilde{w}$ characterized by the super–contact condition $v''(\widetilde{w}) = 0$, and Equation (4.10) becomes

$$rv(w) = \mu + \rho w v'(w) + \frac{\beta^2}{2} v''(w), \tag{4.14}$$

which is a well–known problem (see DS). The boundary condition at $w = w_m$ follows directly from Equation (4.4), since

$$\Pi = \overline{X} v\left(\frac{W_m(\overline{X})}{\overline{X}}\right) = \overline{X} v(w_m). \tag{4.15}$$

This results in $v(w_m) = \Pi/\overline{X}$. By now we have a fair description of the problem, which we formalize below:

Proposition 2 *The function v is the unique solution on $(w_m, \tilde{w})$ to the differential equation*

$$rv(w) = \mu + \rho w v'(w) + \frac{\beta^2}{2} v''(w)$$

subject to the boundary conditions $v(w_m) = \Pi/\overline{X}$ and $v'(\tilde{w}) = -1$. The mapping $w \mapsto v(w)$ is strictly concave on $[w_m, \tilde{w}]$ and the size–adjusted payment barrier is characterized by the super–contact condition $v''(\tilde{w}) = 0$.

Furthermore, directly from this Proposition we have the following result:

Corollary 1 *The relation $w_m < \widetilde{w}$ holds in equilibrium.*

Hence the odd configuration in which termination (w_m) and payment $(\widetilde{w})$ are in conflict cannot arise in equilibrium. In Figure 2 we show the scale–adjusted value function $v(\cdot)$ for parameter values $\mu = 0.6$. $\beta = 0.8$, $r = 0.25$, $\rho = 0.3$, $\tilde{w}_m = 0.7$, $\Pi = 0.414$ and $\overline{X} = 0.4$. The resulting continuation value at the upper boundary $\tilde{w}$ equals 2.349.
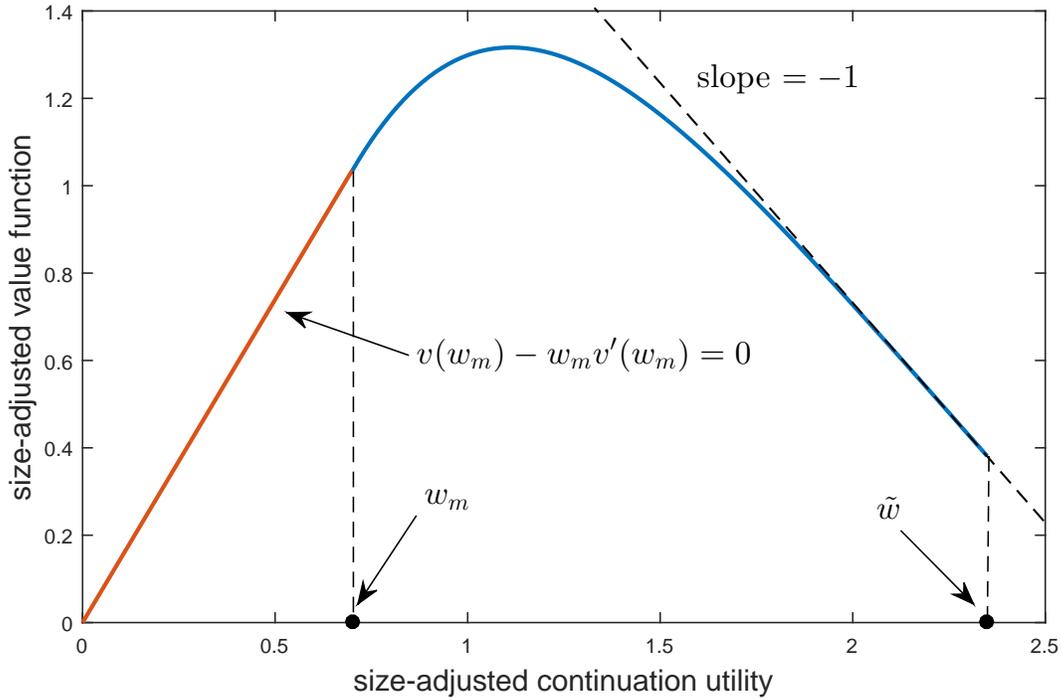
Figure 2: Size–adjusted value function and the reflecting barriers

If we define $\overline{x}_t := \overline{X}/X_t$, the stopping time τ becomes

$$\tau := \inf\left\{t \geq 0 \middle| w_t = w_m, \overline{x}_t = 1\right\}. \tag{4.16}$$

It is not complicated to show that for any two dates $s > t$, it holds that $\mathbb{P}\{w_s = w_m | w_t\} > 0$, which is simply a consequence of the Brownian stochastic driver in the dynamics of w. It is trickier to prove, though, that $\overline{x}$ almost–surely reaches one (i.e. termination) in finite time.

Proposition 3 *The stopping time defined in Expression (4.16) satisfies* $\mathbb{P}\{\tau < \infty\} = 1$.

4.3 Solution and optimal contract

Once we have found v, we can recover V using the homogeneity Property (4.9). In particular, the payment barrier is given by

$$\widetilde{W}(X_t) = \widetilde{w}X_t \tag{4.17}$$

and, along the boundary $W = w_m X$,we have

$$V(X, w_m X) = \Pi X / \overline{X}.$$

We are now in the position to formally state and prove our conjecture regarding the optimal values for the agent's sensitivities to the volatility of cash flows and to large losses.

Proposition 4 *For any size X_t the sensitivity of the agent's continuation value and the penalty should be set as low as possible: $\beta_t \equiv \beta$ and $P_t = w_m X_t$.*

The first part of this claim is well known and follows directly by inspection of Equation (4.14), where the mapping $w \mapsto v(w)$ is known to be concave – so β should be as small as possible. Exposing the agent to risk is costly to the principal, who does it only just enough to generate the right incentives. The second part is new. Setting the penalty level any higher than $W_m(X_t)$ does not affect the incentive Constraint (3.8). Furthermore, $W_m(X_t)$ is an intervention threshold: setting P_t higher that $W_m(X_t)$ only increases the probability of intervention. Whether downsizing or termination, intervention is costly to the principal; thus, its probability of occurrence should be minimized (subject to incentive compatibility). Last, downsizing is preferable to termination for size X_t larger than $\overline{X}$. From the homogeneity Property (4.9) and Proposition 2 we may derive the structure of the value function V.

Theorem 1 *When $\widetilde{W}(X) < W_m(X)$, there exists a unique solution $V(\cdot, \cdot)$ to Equation (4.3) together with the boundary conditions*

$$V_W(X, \widetilde{W}(X)) = -1 \text{ and } V(\overline{X}, W(\overline{X})) = \Pi.$$

The payment barrier $\widetilde{W}(X)$ equals $\widetilde{w}X$ and the downsizing one corresponds to the line $W = w_m X$. For any $X \in [\overline{X}, X_0]$ the mapping $W \mapsto V(X, W)$ is concave and there exists a maximal liquidation scale X^* such that if we set $\overline{X} = X^*$ then the mapping $X \mapsto V(X, W)$ increasing for any $W \in [w_m X, \widetilde{W}(X)]$. This corresponds to the condition $v(w_m) - w_m v'(w_m) = 0$. Therefore the contract starts with initial value (X_0, W_0), where W_0 is the maximizer of $V(X_0, W)$. Finally the cumulative transfers to the agent are computed as

$$I_t = \int_0^s X_s di_s, \text{ where } i_s = \max\left\{0, \max_{0 \le s \le t}\left\{w_s - \overline{w}\right\}\right\}.$$

To conclude our analysis we note that termination at $\overline{X}$ implies that $W_m(\overline{X}) = w_m \overline{X} > 0$ while the agent's reservation value is 0. This difference may be interpreted as a rent the agent receives because of the risk taking problem, or could be extracted by imposing an "entry fee" in case the agent owns some assets (cash). In terms of implementation this entry fee is simply the purchase price of the equity she is made to hold.

4.4 Comparative statics

The simplicity of the differential Equation (4.14) allows for the derivation of accessible comparative statics. We do not present all the possible comparative statics results so as to not repeat DS, but rather put emphasis on the newer ones. In particular we focus on results pertaining the incentive constraint $w_m \geq \frac{\beta}{\sigma} \frac{\Delta \mu}{\lambda}$.

Said analysis is not trivial: the function $v(\cdot)$ and the upper boundary $\widetilde{w}$ are jointly determined and are parametrized by the lower boundary w_m. Therefore, changing a parameter at the lower boundary affects it directly, as well as the solution $v(\cdot)$ and the upper boundary $\widetilde{w}$. That is, we need to understand the behavior of $v(\cdot)$ to understand the effects of parameter changes on the upper boundary $\widetilde{w}$. Heuristically, from the differential Equation (4.14) we can write $v(w)$ along the equilibrium path as

$$v(w) = \mathbb{E}\left[\int_0^\tau e^{-rs}\left(\mu + wv'(w) + \frac{\beta^2}{2}v''(w)\right)ds + e^{-r\tau}\pi|w\right]$$

and then differentiate with respect to any parameter. For example, a marginal increase in the drift yields

$$\frac{\partial v}{\partial \mu} = \mathbb{E}\left[\int_0^\tau e^{-rs}\right].$$

That information can then be used to totally differentiate the boundary condition

$$rv(\widetilde{w}) + \rho\widetilde{w} = \mu$$

to obtain

$$\frac{\partial \widetilde{w}}{\partial \mu} = \frac{1}{\rho - r} - \frac{r}{\rho - r}\frac{\partial v}{\partial \mu},$$

and straightforward computations show that $\frac{\partial \widetilde{w}}{\partial \mu} > 0$.

Proposition 5 *For any $w \in [w_m, \tilde{w}]$, the principal's size–adjusted value $v(w)$ is (*i*) decreasing in the volatility σ, (*ii*) decreasing in the diversion efficiency η, (*iii*) decreasing in the return to risk taking $\Delta\mu$, (*iv*) increasing in the jump intensity λ.*

*The value of the barrier $\tilde{w}$ (*i*) increases with the volatility σ, (*ii*) increases with the diversion parameter η, (*iii*) increases with the return to risk taking $\Delta\mu$ (*iv*) decreases with the jump intensity λ.*

Some of these results deserve comments. That the value $v(w)$ decreases in the volatility of cash flows is not surprising. When returns are more volatile the agent must be offered more of the cash flow: $\beta \equiv \eta\sigma$. Likewise with the diversion efficiency η: when it is easier to divert cash, more must be handed over to the agent so as to prevent her from doing so. In fact, $v(w)$ decreases in η on two accounts; first, more must be given to the agent to deter diversion. Second, increasing η directly increases

$$w_m \equiv \eta\frac{\Delta\mu}{\lambda}.$$

In other words, it shifts the boundary and therefore affects the value of $v(w)$ for any w. The intervention threshold w_m is likewise increasing in $\Delta\mu$ and decreasing in λ. Therefore, inefficient termination is also triggered more frequently when $\Delta\mu$ increases, but less so when λ increases. This latter claim may be counterintuitive at first; however, one must bear in mind that losses only arise off the equilibrium path. A higher intensity λ induces a higher expected penalty λP_t for the agent (for a given P_t); hence when λ increases, said penalty may be lowered, which then reduces the likelihood of termination.

Parameters increasing (decreasing) the termination threshold w_m also increase (decrease) the barrier $\tilde{w}$. With the parameter η it is almost obvious as it acts like σ. For the remaining ones, increasing w_m decreases $v(w)$ for all w, so that the point $\tilde{w}$ characterized by $v'(\tilde{w}) = 1$ and $v'(\tilde{w}) = 0$ lies further out. The intuition is that when preserving the risk taking condition is more difficult, the principal is more reluctant to disburse cash and prefers increasing the agent's continuation value w.

5 Implementation

We suggest two practical implementations of the optimal contract. First we show that we can implement it using standard securities as in DS, but with more instruments. Our optimal contract maps remarkably well into real–life financial arrangements. We also compute security prices. The second implementation is one of financial regulation, which is also faithful to what one observes in practice.

5.1 Capital structure

If we suppose that the principal represents the financiers of a firm who contract with a CEO (the agent), the contract may be implemented, as in DS, with a mix of equity, debt, a credit line and a dividend policy to which we must add some critical covenants: the agent must accept the downsizing process X^d. When W reaches $\widetilde{W}$ dividends are payed out to equity holders. On the other hand, when W reaches W_m the *principal* retains the right to downsize. As a consequence, all other (debt) contracts are also contingent on size. We interpret this as a contractual covenant.

The credit line serves a dual purpose: first, it keeps track of the agent's size–adjusted continuation value, as in DS. Second, it is the device that activates downsizing. In this sense, it matches quite well the characteristics of a revolving credit. Debt amounts to

$$rD = X_t \left(\mu - \rho \frac{w_m}{\eta} - \rho c \right),$$

which also has to be flexible. When managing leverage, as the contract dictates, one must be able to manage the size of the debt. The first term within the brackets corresponds to the gross cash flows, the second one is the liquidation value to the agent (when incentive compatibility can no longer be guaranteed) and the last one is the limit on the credit line. The latter is characterized by

$$M_t = \left(X_t - \overline{X} \right) \left(c - \frac{w_t - w_m}{\eta} \right) \quad \text{and}$$
$$c = \frac{\widetilde{w} - w_m}{\eta}.$$

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The first line is the flow value of the balance M_t as a function of the current size X_t and the continuation value w_t. At w_m the balance is $M_t = c\left(X_t - \overline{X}\right)$: the credit line is fully drawn, which triggers downsizing and results in a lower limit, so a lower balance. When X_t reaches $\overline{X}$ the balance on the credit line is zero, as is the limit, and the firm is terminated. The second line determines the limit on the credit line – in size-adjusted terms. Hence the credit line enforces covenants in the form of

1. a leverage ratio: $w_t \geq w_m$ and

2. a downsizing process X^d

that correspond to the risk taking constraint and define a resolution mechanism when it is breached. We also see that the ability to downsize plays a critical role. Absent appropriate downsizing, the balance M_t may suggest that the credit line is fully drawn, which belies the true level of the agent's continuation value and the options available to the principal.

Our next corollary illustrates the usefulness of the abstract analysis carried out in Sections 3 and 4. Only some of the the results of Proposition 5 can be directly exploited to analyze the behavior of the securities used to implement the contract. Hence, the observation of securities data may be insufficient to draw conclusions regarding what drives their behavior.

Corollary 2 *The debt level D is increasing in the intensity λ and decreasing in the gain $\Delta\mu$; the credit line c is ambiguous. Comparative statics of D and c with respect to η are ambiguous.*

The ambiguous result on the credit limit c stems from the fact that both $\widetilde{w}$ and w_m enter its definition, and that both are affected by the parameters of interest. The debt level increases when the penalty for risk taking is more effective (when λ increases). In such case the agent may hold less equity and the same project may, thus, be financed with more debt instead. The converse holds when risk taking is more attractive ($\Delta\mu$ decreases).

Remark 2 *In Biais and Casamatta (1999) the optimal contract is also implemented using debt and equity. As in this paper, equity is used to control the risk–shifting problem and*

debt addresses the effort problem. When the effort problem is dominant, options are required in Biais and Casamatta (1999), i.e. free securities that only pay in the high state. Here, our continuation value w and payment policy dI_t play this role.

The face value of the debt D and the credit line $C(X_t) = c(X_t - \overline{X})$ may also be simply computed. Substituting for c in D one has

$$rD = X_t \left(\mu - \rho \frac{w_m}{\eta} - \rho c \right)$$
$$= X_t \left[\mu - \rho \left(\frac{\Delta \mu}{\lambda} + \frac{\widetilde{w} - w_m}{\eta} \right) \right]$$
$$= \frac{X_t}{\eta} \left[\mu(\eta - 1) + rv(\widetilde{w}) \right]$$

where the last line uses the boundary condition $rv(\widetilde{w}) + \rho \widetilde{w} = \mu$. Therefore,

$$D = \frac{X_t}{\eta} \left[\frac{\mu}{r}(\eta - 1) + v(\widetilde{w}) \right]$$

and in the special case where $\eta = 1$, $D = X_t v(\widetilde{w})$. The face value of the debt, thus, increases with the size X_t. Likewise for the credit line, we find

$$c = \frac{\widetilde{w} - w_m}{\eta}$$
$$= \frac{1}{\eta} \left(\frac{\mu - rv(\widetilde{w})}{\rho} \right) - \frac{\Delta \mu}{\lambda}$$

and c also depends on the value of the firm: a valuable firm (say, with a very profitable project) needs little revolving credit but may carry a large debt. Finally we see that w_m has a natural interpretation as an equity requirement: the extent of any debt holding D or $C(X)$ is decreased by w_m. Whether the debt is risky hinges on whether $\Pi \geq (<)D$ – see DS for details.

5.2   Regulation contract

We contemplate a regulation contract where the principal is a regulator and the agent a regulated financial institution. Let y_t denote the firm's equity, of which a fraction η is awarded to the agent: $w_t = \eta y_t$, so in particular $w_m = \eta y_m$. The fraction $1 - \eta$ may

represent outside equity held by a diffuse investor base, or a fraction of ownership retained by the government. Bank debt is made of demand–deposit accounts, hence it has to be kept flexible

$$rD = X_t \left(\mu - \rho \frac{w_m}{\eta} \right) = X_t \left(\mu - \rho y_m \right),$$

which asserts that cash flows remunerate the debt, net of the flow value of the agent's termination payoff.

Unlike the previous implementation, here there is no credit line that may be used as a substitute for observing the continuation value of the agent. However, since the firm's equity is linear in said continuation value, one may keep track of the value of the firm instead. Given that

$$W_t = \int_t^\tau e^{-\rho(s-t)} dIs + e^{-\rho\tau} W_m,$$

the market value of equity is

$$Y_t := \int_t^\tau e^{-\rho(s-t)} \frac{dIs}{\eta} + e^{-\rho\tau} \frac{W_m}{\eta} \tag{5.1}$$

Furthermore, the accounting identity of the balance sheet reads

$$D_0 + Y_0 = A_0 + G_0$$

where A_0 denotes assets, i.e. loans, and G_0 is *goodwill*. The book–to–market ratio is defined as

$$B/M := \frac{Y_0}{Y_t} = \frac{A_0 + G_0 - D_0}{Y_t};$$

hence, a leverage ratio implementing the constraint $w_t \geq w_m$ amounts to

$$y_t \geq y_m \text{ that is, } y_0 \geq y_m M/B.$$

Using mark–to–market accounting we have that $B/M = 1$, which yields the following result:

Proposition 6 *The regulatory contract may be implemented with a leverage ratio*

$$y_0 \geq \frac{w_m}{\eta} = \frac{\Delta\mu}{\lambda}$$

based on the book–value of equity Y_0 defined in Expression (5.1), provided mark–to–market accounting is used, together with the downsizing process X_t^d and the dividend payout policy characterized by the process I.

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The value y_0 of the leverage ratio based on the book–value of equity is an intervention threshold, together with a downsizing policy. Here the equity requirement is not designed as a buffer against losses, which are too large anyway. Rather, it is a preemptive threshold. The regulatory contract goes as far as prescribing

- that the agent (say, the CEO) holds enough equity;

- that mark–to–market accounting is used;

- a dividend policy; and

- that the agent does not control the size of the firm.

We note that the use of mark–to–market accounting and restrictions on dividend policies accord well with current practice. In Australia, for instance, the local regulator goes as far as regulating some aspects of executive compensation in banks. More generally, the last point suggests that the allocation of control on the size of a firm is an important aspect or corporate governance.

6 Extensions

6.1 Upsizing: costly investment

In our main analysis we only consider downsizing, i.e. $(X_t, t \geq 0)$ is a non–increasing process. Clearly, however, firms also grow, and given that dS_t is linear in X_t, the principal may sometimes find it profitable to invest in the firm. In this section we look precisely into that option. To do so, we restrict the size X_t of the firm to be at most the initial size X_0. That level is not essential – it may be interpreted as some optimal size, as we also show below. For technical reasons, though, it is important that X_t remains bounded.

6.1.1 Proportional investment costs

We introduce the process $(g_t, t \geq 0)$ of investment rate, which is naturally (and strictly) bounded by the principal's discount rate: there exists $\overline{g} \in [0, r)$ such that $g_t \in [0, \overline{g}]$ for all

$t \geq 0$. Intuitively speaking, if $g_s \equiv g > 0$ for $s \in (t, t+dt)$ then $X_{t+dt} = (1+g)X_t dt$. Investing carries the unitary cost c. With a positive investment cost, there is never simultaneous downsizing and investment; therefore, the process X remains of bounded variation and, in fact, continuous. Hence, the arguments regarding the absence of cross–variation terms in Section 4 remain valid. As a consequence, the HJB equation becomes

$$rV(X_t, W_t)dt = \mu X_t dt + \sup_{dX_t, dI_t, g_t} \left\{ -dI_t + V_W(X_t, W_t)(\rho W_t dt - dI_t) + V_X(X_t, W_t)dX_t \right.$$
$$\left. + \frac{\beta^2 X^2}{2} V_{WW}dt + g_t X_t \left(V_X(X_t, W_t) - c\right)dt \right\}$$

where the last term represents the instantaneous benefit of increasing the venture's size. It is precisely through the factor $V_X(X_t, W_t) - c$ that we determine the investment region. Notice that investing has no bearing on the agent's incentives. In size–adjusted terms, the HJB equation becomes

$$rv(w_t)dt = \mu dt + \sup_{di_t, dx_t, g_t} \left\{ -di_t + v'(w_t)(\rho w_t dt - di_t) + (v(w_t) - w_t v'(w_t))dx_t \right.$$
$$\left. + \frac{\beta^2}{2} v''(w_t)dt + g_t \left(v(w_t) - w_t v'(w_t) - c\right)dt \right\}, \tag{6.1}$$

still subject to Constraints (3.7) and (4.11). As noted in BMRV a necessary condition for investment to ever take place is

$$v(\widetilde{w}) + \widetilde{w} > c, \,^9$$

that is, the maximum of the (unit) social value of the firm must exceed the unit cost of investment. We suppose this holds. From the HJB Equation (6.1) investment only takes place if

$$v(w) - wv'(w) > c,$$

at which point the linearity of the investment return implies that g_t is as high as possible. The choice of $\overline{X} = X^*$ as in Theorem 1 guarantees that $v(w_m) - w_m v'(w_m) = 0$, which yields the following result:

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[9]The equality may be ignored.

Proposition 7 *Suppose the principal may invest in the firm. Then investment takes place only when the agent's continuation reaches the threshold w_i characterized by the condition*

$$v(w_i) - w_i v'(w_i) = c; \qquad (6.2)$$

at that point investment takes place at the maximum rate. In other words, whenever $w_t \geq w_i$ we have $g_t = \bar{g}$. Furthermore, $w_m < w_i < \widetilde{w}$ and the value function is smooth at w_i.

The investment threshold w_i is strictly interior. Obviously it cannot coincide with the downsizing barrier w_m (investing is costly). When the principal invests, he delays payments to the agent $(\widetilde{W} = X_t \widetilde{w})$; it is as if everyone becomes more patient as one observes in Equation (6.1). Hence, while costly, investment reduces the cost of delaying payments; increasing the size pushes out the payment barrier $\widetilde{W}$. However it also moves the boundary W_m (also linearly) and so it opens the door to future downsizing. However, thanks to the concavity of $v(\cdot)$, w_i is strictly bounded away from w_m. Thus, investing requires a history of good news after any downsizing, not to make any inference about the agent (none need be made) but, rather, to ensure there will be no downsizing for some time to come. We can complement this result with

Corollary 3 *The investment threshold w_i*

1. *increases with (*i*) the volatility σ, (*ii*) the diversion parameter η, (*iii*) the risk taking benefit $\Delta\mu$;*

2. *decreases with the intensity λ.*

The proof is immediate (and therefore omitted) as soon as one considers that increasing the no–risk–taking threshold w_m is accompanied by a uniform decrease in $v(\cdot)$ (see Equation (A17) in the proof of Proposition 5). Since the condition

$$v(w_i) - w_i v'(w_i) = c \qquad (6.3)$$

for investment is relevant on the increasing range of $v(\cdot)$, it can only be satisfied at a higher level w_i. Worsening the moral hazard problem in the sense of higher diversion benefits

η or risk taking benefit $\Delta\mu$ is deleterious for investment. That is, the principal "delays" investment for longer because the intervention threshold w_m is more demanding when the risk–taking incentives are worse so future downsizing is more likely.

6.1.2 Lumpy investment: fixed cost

Investing may entail frictions such as costly fund raising, which is documented to feature fixed costs, like the installation of fixed–size assets (machinery, office space...) in addition to the marginal cost c. In the sequel we denote the fixed investment cost by F and by ΔX the change in firm size following an investment decision.[10] For (X, W) given, an investment decision is made so as to maximize

$$\underbrace{V(X + \Delta X, W) - V(X, W)}_{\text{increment in principal's value}} - \underbrace{(c\Delta X + F)}_{\text{invest. costs}}.$$

Provided the mapping $\Delta X \mapsto V(X + \Delta X, W) - c\Delta X$ is quasi–concave, the first–order conditions for optimality is

$$V_X(X + \Delta X, W) = c,$$

which is akin to Expression (6.3). Assuming that $V_{XX} \neq 0$ always holds[11] we may invoke the Implicit Function Theorem (IFT). Then, there exists a function $\xi : \mathbb{R} \to \mathbb{R}$ such that $V_X(\xi(W), W) = c$, and ΔX satisfies

$$\Delta X = \xi(W) - X. \tag{6.4}$$

The magnitude of any upsizing is contingent on the agent's current continuation utility, as before, and the firm's current size. As a consequence of Expression (6.4), upsizing is lumpy – precisely because of the fixed investment costs. Fixing $X \in (\overline{X}, X_0)$, the principal chooses to increase the firm's size if W satisfies

$$\nu(X, W) := V(\xi(W), W) - V(X, W) - c(\xi(W) - X) - F = 0. \tag{6.5}$$

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[10]Given the presence of the time–independent quantities F and ΔX, homogeneity of the value function $V(X, W)$ in X does not carry over. A thorough study of the principal's value function would require the use of new methods beyond the scope of the current paper; hence the presentation below is mathematically not as rigorous as the rest of the paper.

[11]In the prequel we had $V_{XX}(X, W) = -wv''(w) > 0$, so this is by no means far–fetched.

Observe that the total derivative

$$\frac{d}{dW}\nu(X, W) = \underbrace{\left[V_X(\xi(W), W) - c\right]\xi'(W)}_{=0 \text{ from the def. of } \xi(W)} + \underbrace{\frac{\partial}{\partial W}\left[V(\xi(W), W) - V(X, W)\right]}_{V_X\xi'(W)>0}.$$

Using the again the IFT, there exists $\omega : \mathbb{R} \to \mathbb{R}$ such that

$$\nu(X, \omega(X)) = 0.$$

Increasing the size X discretely may violate incentive compatibility. Hence we must require that, at the time of upsizing, W_t satisfies $W_t \geq w_m(X_t + \Delta X)$; in other words

$$\omega(X) \geq w_m \xi(W).$$

Finally, notice that the principal is still subject to the usual incentive Constraints (3.7) and (4.11). In summary,

Proposition 8 *Consider the fixed– and proportional–cost investment problem. The principal's value function $V(\cdot, \cdot)$ is the solution to the differential equation*

$$rV(X, W) = \mu X + \rho W V_W(X, W) + \frac{\beta^2 X^2}{2} V_{WW}(X, W)$$

on the domain delimited by the graph of $W = \omega(X)$ and the inequalities $W \leq \widetilde{W}(X)$ and $X \in [\overline{X}, X_0]$ with boundary conditions

$$V_W(X, \widetilde{W}(X)) = -1, \quad V(\overline{X}, w_m\overline{X}) = \Pi \quad and \quad \nu(X, \omega(X)) = 0.$$

For each $X \in (\overline{X}, X_0)$ upsizing occurs whenever $W_t = \omega(X)$ and it is of size $\Delta X = \xi(W) - X$.

6.1.3   Endogenous bound on size

Hitherto we have worked under the assumption that, at any date $t \geq 0$, the scale of the firm is bounded above by its initial size X_0. When the investment cost is linear, as in Section 6.1.1 this bound is necessary to guarantee that the process X is of bounded variation, but its exogeneity is not particularly restrictive.

Suppose instead that the investment cost is a convex function $C(\cdot)$ of the investment $g_t X_t dt$ with $C', C'' > 0$. Following the logic of Section 6.1.1, the HJB equation dictates that investment should take place whenever

$$g_t X_t V_X \geq C(g_t X_t), \text{ or } V_X \geq \frac{C(g_t X_t)}{g_t X_t},$$

where $V_X = v(w) - wv'(w)$ is independent of X_t. By the convexity of $C(\cdot)$, there exists a size X_t such that

$$\frac{C(g_t X_t)}{g_t X_t} > c,$$

for any $c < \infty$. Therefore, the investment condition

$$v(w_i) - w_i v'(w_i) = \frac{C(g_t X_t)}{g_t X_t}$$

eventually pins down a finite size.

6.2 No liquidation at the boundary

Thus far we simplified the analysis by imposing a termination condition with an exogenous liquidation value at the lower boundary. In many cases, however, liquidation may not be desirable or even possible. For example, liquidating Lehman Brothers proved to be very disruptive and socially costly. Likewise, a large utility company (e.g. electricity transmission) can hardly be liquidated; ditto for a clearinghouse.

In such cases, continuation of service prevents liquidation. Instead, the contract must be terminated, the agent replaced and a new contract with a new agent must be initialized. This is a costly endeavor that we model by introducing the fixed restructuring cost $K > 0$. Assuming that the principal would aim at restructuring the firm so as to have its scale back to X_0, the total benefit of restructuring reads

$$V(X_0, W_0^*) - V(\overline{X}, W_m) - (X_0 - \overline{X}) - K,$$

where W_0^* maximizes the mapping $W \mapsto V(X_0, W)$. This has to be contrasted with Π, the benefits of liquidation. In this case we would have that the boundary condition at $(\overline{X}, W_m)$ is

$$V(\overline{X}, W_m) = \max \left\{ \Pi, V(X_0, W_0^*) - (X_0 - \overline{X}) - K \right\}. \tag{6.6}$$

The above condition introduces a fixed–point type problem, since $V(\cdot, \cdot)$ appears on both sides of Expression (6.6). If K is not too large (relative to μ) then the principal opts for the restructuring option, whereas too–high restructuring costs would lead to termination. We may again make use of the homogeneity property of the value function and state the size–adjusted version of the aforementioned problem as

$$rv(w) = \mu + \rho w v'(w) + \frac{\beta^2}{2} v''(w) \tag{6.7}$$

subject to $v'(\widetilde{w}) = -1$ and $v''(\widetilde{w}) = 0$ and satisfying

$$v(w_m) = \max\left\{\pi, x_0 - 1 - k + x_0 v(w^*)\right\}, \tag{6.8}$$

where w^* again maximizes $v(\cdot)$ over $[w_m, \tilde{w}]$, $\pi := \Pi/\overline{X}$, $x_0 := X_0/\overline{X}$ and $k := K/\overline{X}$. Observe that now for each choice of $\tilde{w}$ we have a solution $v(\cdot; \tilde{w})$ and the choice of $\tilde{w}$ is made in such a way that the boundary condition is satisfied. We formalize this argument in the following result:

Proposition 9 *There is a maximal $\overline{k}$ such that if $k \leq \overline{k}$ then the restructuring problem has a unique solution $v(\cdot)$ that solves Equation (6.7) subject to $v'(\widetilde{w}) = -1$ and $v''(\widetilde{w}) = 0$, where $\tilde{w}$ is chosen so that the boundary condition $v(w_m) = x_0 - 1 - k + x_0 v(w^*)$ is satisfied.*

7   Discussion

Leverage and scale.   The term $w_t = W_t/X_t$ may be interpreted in several ways. In the context of executive compensation, it could be understood as the stake(s) of the executive(s) in the firm. If so, then this model suggests that said stake must be significant enough compared to the wealth of the agent and it must increase in the size of the firm.[12]

An alternative interpretation, which we suggest when implementing the contract, is that of leverage. In such case W_t maps to the equity market–value of the firm and the model (i) provides foundations to leverage regulation and roots its origin squarely in the opportunity to speculate and (ii) it shows that a firm has incentives to gamble for resurrection when the

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[12]In this model the agent's continuation value is her only wealth.

market value of its equity is too low. Indeed, as argued by Diamond and Rajan (2000) in the context of banking, equity is *not required* to buffer credit losses. Deposit holders (and other kinds of creditors) may be shielded from credit losses by subordinated debt; what matters is that claims be junior to deposits. Here we show that equity is *necessary* to prevent excessive risk–taking.

The model also predicts that intervention is necessary along the equilibrium path, where it takes the form of downsizing. This maps cleanly into divestment by firms and was widely observed in the recent GFC, e.g. AIG selling assets like Hartford Steam Boiler, 21st Century Insurance, Transatlantic Re or ALICO to pay debt off, i.e. to reduce its leverage. Likewise, early in the GFC, Merrill Lynch sought to sell its commercial finance business to GE.

Monitoring. Monitoring is a standard remedy to moral hazard. Here one has to be careful in regards to *what* should be monitored. Monitoring that somehow results in reducing the change $\Delta\mu$ in the drift is uniformly positive: it reduces w_m by curtailing the incentives to engage in risky activities. However, monitoring to reduce the probability λ of catastrophic losses is uniformly bad(!); it increases w_m and decreases both $v(\cdot)$ and $\widetilde{w}$. It is a license to speculate: a large loss is even less likely. Hence, under the lens of system–wide risk management, it is better to reduce the magnitude of losses (L) than their frequency λ.

Large losses and risk taking. In the main text we restrict our attention to contracts in which the agent is never allowed to take excessive risk along the equilibrium path. More generally one could conceive of instances in which the principal allows for risk taking; this may be optimal if said losses are not too large. This is "gambling for resurrection" for the principal.

For the agent to proceed in such fashion, the contract with $P_t = \frac{\beta}{\sigma}\frac{\Delta\mu}{\lambda}X_t$ must clearly be relaxed. Given that L is fixed and the mapping $w \mapsto v(w)$ is increasing it is immediate that the optimal $P_t \equiv 0$. That is, there is no object punishing the agent for speculating at lower values of $v(\cdot)$ if one allows for speculation at higher values. We also need to distinguish two regimes depending on how creditors are ranked in the case of failure and liquidation. First, if the agent ranks after other creditors – as in bankruptcy proceedings in the United States

– she receives nothing when the firm is wiped out by losses. In this case, when the agent is allowed to speculate her continuation value follows

$$dW_t = \rho W_t dt - dI_t + \beta dZ_t - \lambda W_t$$
$$= (\rho - \lambda)W_t dt - dI_t + \beta dZ_t.$$

Assume $\rho > \lambda$, otherwise the agent never wants to speculate. In the second case, the agents may rank before any other creditors (as in much of continental Europe). Let us assume, for the sake of simplicity, that the agent enjoys full recovery. Then, her continuation value is unaffected in case of large losses:

$$dW_t = \rho W_t dt - dI_t + \beta dZ_t.$$

It is intuitive then that, for values of W_t below some threshold $\widehat{W}$, the agent will choose to take risks and otherwise not. Beyond this the analysis becomes quite involved and and we postpone it for future research. To see why, note that now losses L may arise in equilibrium and so enter the HJB equation for a range of values of W_t. Dividing by X_t one has $L/X_t := l_t$, which depends on t; therefore the value $\widehat{w}_t := \widehat{W}/X_t$ depends on both time t and on l_t.

Alternative Poisson process $N(a)$. We have adopted a stark–losses process with $\lambda(1) = 0$; as a result there are no losses along the equilibrium path. In contrast, BMRV take $\lambda(1) > 0$ and $\lambda(0) = \lambda(1) + \Delta\lambda$: there are always losses, even in equilibrium, but risk–mitigation is helpful.

With such a technology, it is not complicated to show that our incentive Constraint (3.8) remains unchanged: only the change $\Delta\lambda$ in the probability of losses matters. The HJB equation that characterizes the value function, however, is altered: it includes a delay term that takes into account that the agent's continuation utility jumps down whenever one such loss arises (if losses are moderate). In BMRV any loss is followed by a discrete–size downsizing of the agent's continuation value to preserve incentive compatibility. Here, instead, the process β_t^a is independent of the size and of the loss process $LdN_t(a)$ and, more importantly, reducing the continuation value after a discrete loss may induce a violation of incentive compatibility. When it comes to the value function, downsizing is only required at the

boundary $W_m(X_t)$; so it may follow a discrete loss, or simply be induced by the Brownian process. The differences are, first, the presence of the Brownian process that affects the continuation value W_t; hence, at any time t, W_t may depart from $\mathbb{E}[W_t|\mathcal{F}_t]$ (in fact this is in general the case). In BMRV the model only features a Poisson process as the stochastic driver; outside of these losses, W_t equals $\mathbb{E}[W_t|\mathcal{F}_t]$. The second difference is that downsizing is used to deter risk taking, not to induce effort.

Rents. In this model we require $w_m > 0$ for incentive reasons even though the agent's outside option is 0, i.e. the agent earns rents. Observationally this corresponds to providing executives with seemingly too generous incentive packages, especially if some cash pay–out is necessary for subsistence. Likewise, it may appear that some banks are made out hold too much equity; here it is justified as the only means to deter risk taking. As we have noted earlier, this ex–post rent may be extracted ex–ante through a participation fee that may take the form of a concession fee or a buy–in cost as in a partnership.

8   Conclusion

This paper proposes a contracting model in continuous time with a scalable arithmetic Brownian process and the option to speculate. Speculation, or excessive risk taking, improves the drift of the stochastic process but also introduces the risk of large losses governed by a Poisson process.

Incentive compatibility requires that the agent has a large–enough continuation value at all times. This justifies equity holding as a contractual or regulatory instrument. Due to the scale effect, said continuation value must exceed a threshold that is linear in the size of the project. This implies that size becomes an instrument for incentive compatibility. Satisfying the risk–taking condition requires downsizing along the equilibrium path, which induces a floating barrier. This justifies leverage regulation as in banking, for example.

We implement the contract using a broad array of instruments, especially covenants that enable the use of the downsizing process. This suggests, importantly for corporate

governance, that *who* controls size in organizations is an important question that seems to have been somewhat neglected so far.

A point of interest that this paper does not address is whether the contract is fully optimal in the *unrestricted* class of contracts. What is characterized here is the optimal incentive compatible contract where risk taking is never engaged in. This, however, comes at an additional cost on the range $[\widetilde{W}, w_m X_t]$ (when $\Delta\mu > \lambda$). It is not immediate that the principal always wants to offer such a contract. This question is left to further research.

A Proofs

Proof of Proposition 1: We deal first with diversion, in the spirit of DS, Sannikov (2008) and He (2009). Using Expression (2.2), we may rewrite the dynamics of W^a as

$$dW_t^a = \rho W_t^a dt + \frac{\beta_t^a}{\sigma}\Big(dX_t^a - \mu(a_t)X_t^a dt\Big) - d\widehat{C}_t - P_t^a\big[dN_t - \lambda dt\big]. \tag{A1}$$

Furthermore, we have that

$$\frac{\beta_t^a}{\sigma}\Big(dS_t^a - \mu(a_t)X_t dt\Big) - d\widehat{C}_t^a = \left[\frac{\beta_t^a}{\sigma} - \eta\right]\left[dS_t^a - d\widehat{S}_t^a\right] + \frac{\beta_t^a}{\sigma}\big[d\widehat{S}_t^a - \mu(a_t)X_t dt\big]. \tag{A2}$$

On the one hand, $\mathbb{E}\big[P_t^a\big[dN_t - \lambda dt\big]\big] = 0$. On the other one, it holds that $\mathbb{E}\big[d\widehat{S}_t^a - \mu(a_t)X_t dt\big] \le 0$, since the agent cannot over–report and will not report a jump. This implies that

$$\mathbb{E}\left[dW_t^a - \rho W_t^a dt\right] \ge 0 \Leftrightarrow \left[\frac{\beta_t^a}{\sigma} - \eta\right]\left[dS_t^a - d\widehat{S}_t^a\right] \ge 0 \tag{A3}$$

and because $dS_t^a - d\widehat{S}_t^a \ge 0$, the latter inequality requires $\frac{\beta_t^a}{\sigma} - \eta \ge 0$, which is our first constraint.

We now turn our attention to risk taking. Recall the definition of the strategies $a \equiv 1$ and $\tilde{a} := \big(a_s = 0, s < t; a_s = 1, s \ge t\big)$. The agent's total utility under strategy $\tilde{a}$ satisfies

$$\begin{aligned}
\widehat{\psi}_t^{\tilde{a}} &= \widehat{\psi}_t^1 + \int_0^\tau e^{-\rho s}\big(d\tilde{C}_s^{\tilde{a}} - d\tilde{C}_s^1\big) \\
&= \widehat{\psi}_0^1 - \int_0^t e^{-\rho s}P_s^a\big[dN_s(a_s) - \lambda(a_s)ds\big] + \int_0^t e^{-\rho s}\beta_s^a dZ_s + \int_0^\tau e^{-\rho s}\big(d\tilde{C}_s^{\tilde{a}} - d\tilde{C}_s^1\big).
\end{aligned} \tag{A4}$$

where the second line exploits the MRT and follows from the fact that $\widehat{\psi}_t^{\tilde{a}}$ is a martingale. Let $\mathbb{P}^{\tilde{a}}$ be the probability distribution induced by the strategy $\tilde{a}$ on the paths of $N(\tilde{a})$. Under $\mathbb{P}^{\tilde{a}}$, Equation (A4) becomes

$$\begin{aligned}
\widehat{\psi}_t^{\tilde{a}} &= \widehat{\psi}_0^1 - \int_0^t e^{-\rho s}P_s^{\tilde{a}}\big[dN_s(\tilde{a}_s) - \lambda(\tilde{a}_s)ds\big] - \int_0^t e^{-\rho s}\lambda P_s^{\tilde{a}}ds \\
&\quad + \int_0^t e^{-\rho s}\beta_s^{\tilde{a}}dZ_s + \int_0^\tau e^{-\rho s}\big(d\tilde{C}_s^{\tilde{a}} - d\tilde{C}_s^1\big).
\end{aligned}$$

In order to guarantee that the strategy $a \equiv 1$ is preferable to $\tilde{a}$, it suffices to to make sure that the drift of the semimartingale $\widehat{\psi}^{\tilde{a}}$ is negative, which holds if and only if

$$\int_0^\tau e^{-\rho s} \left(d\tilde{C}_s^{\tilde{a}} - d\tilde{C}_s^1 \right) \leq \int_0^t e^{-\rho s} \lambda P_s^{\tilde{a}} ds. \tag{A5}$$

for an arbitrary time t. If the agent chooses to speculate he may divert $\eta \Delta \boldsymbol{\mu} X_t$ or he may report truthfully to earn a higher payment. In the first case Expression (A5) implies that

$$\eta \Delta \boldsymbol{\mu} X_t \leq \lambda P_t,$$

is sufficient, whereas in the second one we have the sufficient condition

$$\frac{\beta_t}{\sigma} \Delta \boldsymbol{\mu} X_t \leq \lambda P_t,$$

which is our second (risk taking) constraint. ∎

Proof of Lemma 1: We have said that downsizing takes place when Constraint 3.8 is violated. One could think that, in principle, the level of the agent's continuation utility could hit $W_m(X)$ and immediately "bounce back up"; thus, there would be no need to downsize the firm at that point. This is in fact not the case. Indeed, away from $\widetilde{W}(X)$, the agent's continuation utility evolves like

$$dW_t = \rho W_t dt + \beta X_t dZ_t. \tag{A6}$$

Let us assume that for some date $\bar{t}$ it holds that $W_{\bar{t}} = w_m X_{\bar{t}}$ and that at that point there is no downsizing. Then, instantaneously the dynamics of W are

$$dW_t = w_m X(\rho dt + \beta dZ_t), \quad W_{\bar{t}} = w_m X \tag{A7}$$

Let us consider the auxiliary process defined via the equation

$$db_t = \rho dt + \beta dZ_t, \quad b_{\bar{t}} = 1 \tag{A8}$$

and define, for $\epsilon > 0$,

$$B(\epsilon) := \inf_{s \in [\bar{t}, \epsilon]} \left\{ b_s \mid W_{\bar{t}} = 1 \right\}. \tag{A9}$$

Using the Cameron–Martin theorem we know there is an equivalent measure $\mathbb{Q}$ under which b is a standard Brownian Motion. Furthermore the infimum of a standard Brownian motion From Chesney et al. (2009), page 147, we have that for any $a \leq 1$,

$$\mathbb{Q}\{B(\epsilon) > a\} = \Phi\left(\frac{-(a-1) + \rho\epsilon}{\beta\sqrt{\epsilon}}\right) - e^{2(a-1)\rho}\Phi\left(\frac{(a-1) + \rho\epsilon}{\beta\sqrt{\epsilon}}\right), \tag{A10}$$

where Φ is the standard normal cumulative distribution function. Letting $a \to 1$ we obtain that for all $\epsilon > 0$

$$\mathbb{Q}\{B(\epsilon) > 1\} = 0,$$

which, since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, implies $\mathbb{P}\{B(\epsilon) > 1\} = 0$. Therefore, for all $\epsilon > 0$ it holds that

$$\mathbb{P}\left\{\inf_{[\bar{t},\bar{t}+\epsilon]}\{W_s|W_{\bar{t}} = w_m X\} > w_m X\right\} = 0,$$

which concludes the proof. ∎

Proof of Lemma 2: Observe that, analogously to Eq. (10) in DS, the increments of the scale–process X are given by

$$dX_t = -\frac{1}{w_m}\max\{w_m X_t - W_t, 0\}.$$

This means that

$$-w_m dx_t = \max\{w_m - w_t, 0\},$$

which in turn implies that

$$-w_m x_t = \max\left\{0, \max_{0 \leq s \leq t}\{w_m - w_t\}\right\}.$$

From Lemma 6.14 in Karatzas and Shreve (1991) we have that $w_m x = (-w_m x_t, 0 \leq t)$ is the unique continuous process such that, added to the process $\widetilde{w}$ whose dynamics are given by the equation

$$d\widetilde{w}_t = \rho\widetilde{w}_t dt + \sigma dZ_t$$

and defining w via $dw_t = d\widetilde{w}_t - w_m dx_t$, results in:

1. $w_t \geq w_m$ almost surely and

2. the measure $-w_m dx$ only carries mass on the set $\{w = w_m\}$:

$$\int_0^\tau \mathbb{I}_{\{w_s > w_m\}}(-w_m dx_s) = 0,$$

for any w–stopping time τ, i.e the non–decreasing process $-w_m x$ only increases on the set $\{w = w_m\}$.

In other words, $-w_m x$ is the local time at level w_m of $\widetilde{w}$; thus, adding $-w_m dx_t$ to its dynamics yields a the process that, by construction, will exhibit an instantaneous reflection at the level w_m, which concludes the proof. $\blacksquare$

Proof of Proposition 2: Any solution to the differential Equation (4.14) may be written as the sum of the particular solution $v \equiv \mu/r$ and one particular solution to the homogeneous equation

$$rh(w) = \rho w h'(w) + \frac{\beta^2}{2} h''(w) \tag{A11}$$

Let us denote by h_0 and h_1 the particular solutions to Equation (A11) that satisfy $h_0(w_m) = 1, h_1(w_m) = 0, h_0'(w_m) = 0$ and $h_1'(w_m) = 1$. Using these *basis functions* we may write

$$v(w) = \frac{\mu}{r} + a_0 h_0(w) + a_1 h_1(w), \ w \in [w_m, \widetilde{w}]$$

for some $\widetilde{w} \geq w_m$. In order to determine a_0 and a_1 we use the boundary conditions $v(w_m) = \pi := \frac{\Pi}{X}$ and $v'(\widetilde{w}) = -1$:

$$\frac{\mu}{r} + a_0 h_0(w_m) + a_1 h_1(w_m) = \pi \Rightarrow a_0 = \pi - \frac{\mu}{r}$$

and

$$a_0 h_0'(\widetilde{w}) + a_1 h_1'(\widetilde{w}) = -1, \Rightarrow a_1 = -\frac{1}{h_1'(\widetilde{w})} \left[1 + \left(\pi - \frac{\mu}{r} \right) h_0'(\widetilde{w}) \right].$$

Therefore

$$v(w) = v(w; \widetilde{w}) = \frac{\mu}{r} + \left(\pi - \frac{\mu}{r} \right) h_0(w) - \left[1 + \left(\pi - \frac{\mu}{r} \right) h_0'(\widetilde{w}) \right] \frac{h_1(w)}{h_1'(\widetilde{w})}, \tag{A12}$$

which is indexed by $\widetilde{w}$, and satisfies the boundary $v(w_m) = \pi$. Next we need to show that for $w_m \geq 0$ given, there exists a unique $\widetilde{w} \geq w_m$ and a unique corresponding function $v(\cdot; \widetilde{w})$ such that $v''(\widetilde{w}; \widetilde{w}) = 0$. To this end we show that the function $h_1(\cdot)$ is strictly increasing for

all $w \geq w_m$. Indeed, if this were not the case, there would exist some $\hat{w}$ such that $h_1'(\hat{w}) = 0$ and $h_1'(w) \leq 0$, $w \in (\hat{w}, \hat{w} + \epsilon)$ for some $\epsilon > 0$. In other words, $\widetilde{w}$ would be a local maximum of $h_1(\cdot)$; thus, $h_1''(\hat{w}) \leq 0$. From the latter and Equation (A11) we obtain that $h_1(\hat{w}) \leq 0$. However, by construction $h_1(\cdot)$ is strictly increasing on $[w_m, \hat{w})$, so that $h_1(\hat{w}) > h_1(w_m) = 0$. This is a contradiction so we must have $h_1'(w) > 0$ for all $w \geq w_m$.

Next we show that the $\widetilde{w}$ that satisfies $v''(\widetilde{w}; \widetilde{w}) = 0$ and the corresponding function $v(\cdot; \widetilde{w})$ are jointly unique. Let us define $\phi(w) := h_0(w)h_1'(w) - h_1(w)h_0'(w)$ and observe that $\phi(w_m) = 1$. Using the boundary condition $v'(\widetilde{w}) = -1$,

$$
\begin{aligned}
\frac{\beta^2}{2} v''(\widetilde{w}) &= rv(\widetilde{w}) + \rho\widetilde{w} - \mu \\
&= \rho\widetilde{w} + (r\pi - \mu)\left(\frac{h_0(\widetilde{w})h_1(\widetilde{w}) - h_1(\widetilde{w})h_0'(\widetilde{w})}{h_1'(\widetilde{w})} \right) - r\frac{h_1(\widetilde{w})}{h_1'(\widetilde{w})} \\
&= \rho\widetilde{w} + (r\pi - \mu)\frac{\phi(\widetilde{w})}{h_1'(\widetilde{w})} - r\frac{h_1(\widetilde{w})}{h_1'(\widetilde{w})} \qquad\qquad (A13)
\end{aligned}
$$

where the second line follows from substituting Equation (A12) and the third one from a simple rearrangement of terms. Now, the boundary–value problem

$$
\begin{aligned}
\phi'(w) &= h_0(w)h_1''(w) - h_1(w)h_0''(w) \\
&= \frac{2\rho w}{\beta^2}\left[h_1(w)h_0'(w) - h_1'(w)h_0(w) \right] \\
&= -\frac{2\rho w}{\beta^2}\phi(w) \qquad\qquad (A14)
\end{aligned}
$$

where the second line uses Equation (A11), together with the boundary condition $\phi(w_m) = 1$ can be solved in closed form:

$$
\phi(w) = \exp\left\{ -\frac{\rho}{\beta^2}(w^2 - w_m^2) \right\}.
$$

Now multiply both sides of Equation (A13) times $h_1'(w)/\phi(w)$ and re–arrange to obtain

$$
\frac{\beta^2}{2} v''(w)\frac{h_1'(w)}{\phi(w)} = \rho w\frac{h_1'(w)}{\phi(w)} - r\frac{h_1(w)}{\phi(w)} + r\pi - \mu.
$$

We want to show that $v''(\widetilde{w}) = 0$, for which we need that the function

$$
\varphi(w) := [\rho w h_1'(w) - rh_1(w)]\frac{1}{\phi(w)}
$$

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satisfies $\varphi(\widetilde{w}) = r\pi - \mu$. Observe that it must hold that $r\pi - \mu < 0$, otherwise there is no object in hiring the agent, and $\varphi(w_m) = \rho w_m > 0$. Hence, it is enough to show that $\varphi(\cdot)$ is strictly decreasing on $[w_m, \infty)$. Differentiating and using Equation (A11) we have

$$
\begin{aligned}
\varphi'(w) &= e^{\frac{\rho}{\beta^2}(w^2 - w_m^2)}\left[\rho w h_1'' - r h_1' + \frac{2\rho w}{\beta^2}(\rho w h_1' - r h_1)\right] \\
&= -e^{\frac{\rho}{\beta^2}(w^2 - w_m^2)} r h_1'(w) < 0,
\end{aligned}
$$

where the last inequality follows from the fact that $h_1' > 0$. Similarly

$$
\begin{aligned}
\varphi''(w) &= e^{\frac{\rho}{\beta^2}(w^2 - w_m^2)}\left[-r h_1'' - \frac{2\rho w}{\beta^2} r h_1'\right] \\
&= -e^{\frac{\rho}{\beta^2}(w^2 - w_m^2)} r \left[h_1'' + \frac{2\rho w}{\beta^2} h_1'\right] \\
&= -e^{\frac{\rho}{\beta^2}(w^2 - w_m^2)} r \frac{2}{\beta^2} h_1(w) < 0
\end{aligned}
$$

where the last line uses Equation (A11) again. Hence $\varphi(\cdot)$ is decreasing and strictly concave on $[w_m, \infty)$, so $\widetilde{w}$ is unique.

The fact that, when $w_m < \tilde{w}$, the mapping $w \mapsto v(w)$ is strictly concave is a result of DS. The condition $v''(\widetilde{w}) = 0$, which guarantees that v and be extended in $\mathcal{C}^2$–fashion to $[w_m, \infty)$ corresponds to the (optimality) super–contact condition in Dumas (1991). ∎

Proof of Corollary 1: The result follows directly from the proof of Proposition 2, where it is shown there exists a unique $\widetilde{w} \geq w_m$ and a unique corresponding function $w(\cdot, \widetilde{w})$ that is a solution. ∎

Proof of Proposition 3: Observe that from Lemma 1 we have that the liquidation event $\{w_t = w_m, \bar{x}_t = 1\}$ coincides with the event $\{W_t = w_m \overline{X}\}$. Observe that for all $t > 0$ it holds that $W_t < \widetilde{W}(X_0) = \tilde{w}$ and $X_t \geq \overline{X}$. Let define a family of processes via the equations

$$
dW_t^{a,b} = a \, dt + b \, dZ_t - dI_t, \quad W_0^{a,b} = \widetilde{W}_0
$$

for $a, b > 0$ $\widetilde{W}_0 > w_m \overline{X}$ independent of a and b and I inducing reflection at the level $W^{a,b} = \tilde{w}$. The first visitation time of $W^{a,b}$ to the level $w_m \overline{X}$ is increasing in a decreasing in b. Therefore, we can bound τ above with the first visitation time

$$
\tilde{\tau} = \inf\left\{t > 0 \big| W_t^{\rho\tilde{w}, \beta\overline{X}} = w_m \overline{X}\right\}.
$$

In particular

$$\mathbb{P}\{\tau < t\} \leq \mathbb{P}\{\tilde{\tau} < t\}. \tag{A15}$$

From Proposition 3.1 in Hu et al. (2012) we know there exist sequences $\{\lambda_n\}_{n \in \mathbb{N}}$ and $\{c_n\}_{n \in \mathbb{N}}$ of positive numbers such that

$$\mathbb{P}\{\tilde{\tau} < t\} = 1 - \sum_{n=1}^{\infty} c_n e^{-\lambda_n t}.$$

Letting $t \to \infty$ yields, together with Equation (A15), the desired result. ∎

Proof of Proposition 4: The function characterized by Equation (4.14) and the boundary conditions $v(w_m) = \Pi/\overline{X}$ and $v'(\tilde{w}) = -1$ is strictly concave on $[w_m, \tilde{w}]$ (see DS) so that $v''(w) < 0$. Therefore, the smaller the coefficient in from of the second–order term is, the larger (in the $\mathbb{L}^{\infty}$ sense) the solution to Equation (4.14) becomes. Hence β_t should be set as small as possible without violating the no–diversion Constraint (3.7), i.e. $\beta_t \equiv \beta$.

The second claim follows from the fact that downsizing is costly to the principal and it is only done for incentive reasons. This results in the fact that it is optimal to set $P_t = w_m X_t$ in Expression (3.9), since

$$\mathbb{P}\{w_s \leq w_m\}$$

is decreasing in w_m. ∎

Proof of Theorem 1: The existence and uniqueness of a solution Equation (4.3) together with the boundary conditions $V_W(X, \widetilde{W}(X)) = -1$ and $V(\overline{X}, W(\overline{X})) = \Pi$ follows from the homogeneity Property (4.9) and Proposition 2, as do the values from the payment and downsizing barriers.

We obtain that for all $X \in [\overline{X}, 1]$ the mapping $W \mapsto V(X, W)$ is concave from the simple observation that $V(X, W) = Xv(W/X)$ and that the mapping $W \mapsto Xv(X/W)$ is concave, since $v(\cdot)$ is a concave function.

The last statement requires us to compute, for any $W \in [w_m \overline{X}, \widetilde{W}(X)]$, the total derivative

$$\frac{d}{dX}V(X, W) = \frac{d}{dX}Xv(W/X) = v(W/X) - \frac{W}{X}v'(W/X) = v(w) - wv'(w),$$

where $w = W/X$. It follows from the concavity of $v(\cdot)$ that the mapping $w \mapsto v(w) - wv'(w)$ is increasing for all $w \geq 0$. Therefore, it suffices to show that $v(w_m) - w_m v'(w_m) =$

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$\Pi/\overline{X} - w_m v'(w_m) \geq 0$ to show that the mapping $X \mapsto V(X, W)$ is increasing for all $W \in [w_m \overline{X}, \widetilde{W}(X)]$. Geometrically, this amounts to showing that the intersect of the tangent line to the graph of $v(\cdot)$ at v_m with the y–axis occurs above the origin. Clearly, the mapping $\overline{X} \mapsto \Pi/\overline{X}$ is decreasing in $\overline{X}$, which suggests that, if the firm's minimal size is small enough, the desired result will hold; however, we should keep in mind that $v'(w_m)$ depends on $\overline{X}$ through the boundary condition at w_m (the value of $v(w_m)$ influences that of $\tilde{w}$). The remainder of the proof is dedicated to formalizing the previous statement. To this end, let us define the following family of functions:

$$
\begin{cases}
v_{\overline{X}}(w) = \frac{\Pi}{\overline{X} w_m} w \text{ if } w \in [0, w_m] \\
r v_{\overline{X}}(w) = \mu + \rho w v'_{\overline{X}}(w) + \frac{\beta^2}{2} v''_{\overline{X}}(w) \text{ if } w \in (w_m, \tilde{w}(\overline{X})) \\
v'_{\overline{X}}(\tilde{w}(\overline{X})) = -1,
\end{cases}
$$

where $\tilde{w}(\overline{X})$ is characterized by the condition $v''_{\overline{X}}(\tilde{w}(\overline{X})) = 0$. Observe that we then need to show the existence of a maximal $\overline{X}$ such that $v_{\overline{X}}$ is concave. In fact, ideally we would like to have that $v_{\overline{X}}$ is of class $\mathcal{C}^1$. This smooth pasting at w_m would requre that

$$
v_{\overline{X}}(w_m+) = \frac{1}{\rho w_m}\left[r\frac{\Pi}{\overline{X}} - \mu - \frac{\beta^2}{2} v''_{\overline{X}}(w_m+) \right] = \frac{\Pi}{\overline{X} w_m}.
$$

In the sequel we use the notation $\mathcal{L}u(w) = \rho w u'(w) + \frac{\beta^2}{2} u''(w) - ru(w)$ and define the parametric family of functions $v(\cdot : \tilde{w})$ as the (unique) solutions to the boundary–value problem

$$
\mathcal{L}v(w; \tilde{w}) + \mu = 0, \ v'(\tilde{w}; \tilde{w}) = -1, v''(\tilde{w}; \tilde{w}) = 0.
$$

Proceeding in analogous fashion as in the proof of Proposition 2 we can get the following auxiliary result:

Lemma A1 *For each $a > 0$ there exists a unique $\tilde{w} > 0$ such that $v(w_m; \tilde{w}) = a$.*

From the above lemma we have that for each choice of $\overline{X}$ there is a unique $\tilde{w}(\overline{X})$ such that $v(w_m; \tilde{w}(\overline{X})) = \Pi/\overline{X}$. We know that $v_{\overline{X}}(w_m) = \frac{\Pi}{\overline{X}}$ and $v'_{\overline{X}}(w) = \frac{\Pi}{\overline{X} w_m}$, so we have to study the behavior of $v'(w_m+; \tilde{w}(\overline{X}))$. To do so, we require the next lemma:

Lemma A2 *The mapping $\tilde{w} \mapsto v(w_m; \tilde{w})$ is decreasing for all $\tilde{w} > w_m$, whereas $\tilde{w} \mapsto v'(w_m; \tilde{w})$ is increasing. Moreover, there is a maximal $\hat{w} > w_m$ such that $v'(w_m; \hat{w}) = 0$ and $v'(w_m; \tilde{w}) > 0$ for $\tilde{w} > \hat{w}$.*

Proof: Substituting the boundary conditions at $w = \tilde{w}$ in the equation $\mathcal{L}v(\tilde{w}; \tilde{w}) + \mu = 0$ yields $v(\tilde{w}; \tilde{w}) = \mu/r - \rho\tilde{w}/r$; hence, $\tilde{w} \mapsto v(\tilde{w}; \tilde{w})$ is decreasing. The fact that the first–order term in $\mathcal{L}$ is multiplied times w has a scaling effect. By this we mean the following: consider $w_m < \tilde{w}_1 < \tilde{w}_2$. Then $v'(w : \tilde{w}_2) > v'(w : \tilde{w}_1)$ for all $w \in [w_n, \tilde{w}_1]$, where the derivatives at the endpoints are the appropriate one–sided ones. This is clearly the case at $w = \tilde{w}_1$, where $v'(\tilde{w}_1 : \tilde{w}_1) = -1$, but, by the concavity of $v(\cdot; \tilde{w}_2)$, we have that $v'(\tilde{w}_1; \tilde{w}_2) > v'(\tilde{w}_2; \tilde{w}_2) = -1$. Moreover, we have that $v(\tilde{w}_1; \tilde{w}_1) > v(\tilde{w}_1; \tilde{w}_2)$. To see this notice that the straight line starting at $(\tilde{w}_1, v(\tilde{w}_1; \tilde{w}_1))$ with slope -1 has the value

$$\frac{\mu}{r} - \frac{\rho}{r}\tilde{w}_1 - \tilde{w}_2 + \tilde{w}_1$$

at $w = \tilde{w}_2$. This, however, is greater than $v(\tilde{w}_2; \tilde{w}_2)$ if and only if $\rho \geq r$ and since $v'(w; \tilde{w}_2) > -1$ for $w < \tilde{w}_2$ the result follows. Notice that $v(\cdot; \tilde{w}_2)$ restricted to $[w_n, \tilde{w}_1]$ corresponds to the solution to the boundary–value problem $\mathcal{L}v(w) + \mu = 0$, $v'(\tilde{w}_1) = v'(\tilde{w}_1; \tilde{w}_2) > -1$ and $v''(\tilde{w}_1) = v''(\tilde{w}_1; \tilde{w}_2) < 0$. The presence of these boundary conditions then guarantees that $v(w; \tilde{w}_2) < v(w; \tilde{w}_1)$ for $w \in [w_m, \tilde{w}_1]$; hence, in particular $v(w_m; \tilde{w}_2) < v(w_m; \tilde{w}_1)$ as well as $v'(w_m; \tilde{w}_2) > v(w_m; \tilde{w}_1)$. Moreover, they also imply that the mapping $\tilde{w} \mapsto v'(w_m; \tilde{w})$ is decreasing, which guarantees the existence of $\hat{w}$. ∎

With Lemmas A1 and A2 in hand, we may proceed to show the existence of a $\overline{X}$ such that $v_{\overline{X}}$ is a $\mathcal{C}^1$ function. We do so in an iterative way. That is to say, set $\overline{X}_1 > 0$ and, using Lemma A1, let $\tilde{w}_1$ be such that $v(w_m; \tilde{w}_1) = \Pi/\overline{X}_1$. Assume that $v'(w_m; \tilde{w}_1) > \Pi/(\overline{X}_1 w_m)$, the complementary case being analogous. Next, choose $0 < \overline{X}_2 < \overline{X}_1$, which increases the left–hand derivative to $v'_{\overline{X}_2}(w_m) = \Pi/(\overline{X}_2 w_m)$ and the value to $v_{\overline{X}_2}(w_m) = \Pi/\overline{X}_2$. We have from Lemma A2 that, in order as to have $v(w_m; \tilde{w}_2) = \Pi/\overline{X}_2$, it must hold that $\tilde{w}_2 < \tilde{w}_1$. If it is still the case that $v'(w_m; \tilde{w}_2) > \Pi/(\overline{X}_2 w_m)$, we choose $\overline{X}_3 < \overline{X}_2$ and repeat. This process of decreasing $\overline{X}_n$ and $\tilde{w}_n$ eventually leads to $v'(w_m; \tilde{w}_n) < \Pi/(\overline{X}_n w_m)$, since $v'(w_m; \tilde{w}_n) \to 0$ as $\tilde{w}_n \to \hat{\tilde{w}}$, whereas $\Pi/(\overline{X}_n w_m) \to \infty$ as $\overline{X}_n \to 0$. We may then relabel $\overline{X}_n$ as $\overline{X}_2$ and

$\tilde{w}_n$ as $\tilde{w}_2$; thus, there is no loss of generality in assuming $v'(w_m; \tilde{w}_2) < \Pi/(\overline{X}_2 w_m)$, We then choose $\overline{X}_2 < \overline{X}_3 < \overline{X}_1$ and obtain the corresponding $\tilde{w}_2 < \tilde{w}_3 < \tilde{w}_1$ such that $v'(w_m; \tilde{w}_3) > \Pi/(\overline{X}_3 w_m)$. By construction, we can keep alternating this procedure to get nested sequences $\{\overline{X}_n\}$ and $\{\tilde{w}_n\}$, which then have limits $\overline{X}^*$ and $\tilde{w}^*$ that satisfy $v'(w_m; \tilde{w}^*) = \Pi/(\overline{X}^* w_m)$ and $v(w_m; \tilde{w}^*) = \Pi/\overline{X}^*$. This concludes the theorem's proof. ∎

Proof of Proposition 5: Proceeding analogously as in Lemma 6 in DS, we have that, using θ as a dummy for one of the model's parameters, the following Feynman–Kac–type representation for $\partial v/\partial\theta$ holds:

$$\frac{\partial v_\theta(w)}{\partial\theta} = \mathbb{E}\left[\int_0^\tau e^{-rs}\left(\frac{\partial\mu}{\partial\theta} + \frac{\partial\rho}{\partial\theta}w_s v'(w_s) + \frac{1}{2}\frac{\partial\beta^2}{\partial\theta}v''(w_s)\right)ds + e^{-r\tau}\frac{\partial}{\partial\theta}\frac{\Pi}{\overline{X}}\bigg|w_0 = w\right].$$

Recall that $\beta = \eta\sigma$. Then we have

$$\frac{\partial v(w)}{\partial\sigma} = \mathbb{E}\left[\int_0^\tau e^{-rs}\left(\eta^2\sigma v''(w)\right)ds\bigg|w_0 = w\right] < 0,$$

by concavity of $v(\cdot)$. Similarly,

$$\frac{\partial v(w)}{\partial\eta} = \mathbb{E}\left[\int_0^\tau e^{-rs}\left(\eta\sigma^2 v''(w)\right)ds\bigg|w_0 = w\right] < 0.$$

.

For the last set of results we use the first and second–order boundary conditions at $\tilde{w}$ to obtain the expression

$$rv(\tilde{w}) + \rho\tilde{w} = \mu$$

and compute its partial derivatives with respect to the generic parameter θ :

$$r\left(\frac{\partial v(\tilde{w})}{\partial\theta} + v'(\tilde{w})\frac{\partial\tilde{w}}{\partial\theta}\right) + \rho\frac{\partial\tilde{w}}{\partial\theta} = 0 \Rightarrow \frac{\partial\tilde{w}}{\partial\theta} = -\frac{r}{\rho - r}\frac{\partial v(\tilde{w})}{\partial\theta} \tag{A16}$$

The task is then to compute $\partial v(\tilde{w})/\partial\theta$. To this end, we denote, as in DS

$$g_\tau(w) := \mathbb{E}\left[e^{-r\tau}\big|w_0 = w\right].$$

Given that the principal's profit remains the same if the agent's continuation value at liquidation increases by dw_m and liquidation value increases by $v'(w_m)dw_m$, the effect of a change in w_m on the principal's profit is

$$\frac{\partial}{\partial w_m}v(w) = -v'(w_m)g_\tau(w). \tag{A17}$$

From the risk taking constraint

$$w_m = \frac{\beta}{\sigma} \frac{\Delta\mu}{\lambda} = \eta \frac{\Delta\mu}{\lambda},$$

it is immediate that $dw_m/d\eta > 0$, $dw_m/d\Delta\mu > 0$ and $dw_m/d\lambda < 0$. Therefore

$$\frac{\partial v(\widetilde{w})}{\partial\lambda} = \frac{\partial}{\partial w_m} v(\widetilde{w}) \frac{dw_m}{d\lambda} = -v'(w_m) g_\tau(\widetilde{w}) \frac{dw_m}{d\lambda} < 0,$$

$$\frac{\partial v(\widetilde{w})}{\partial\eta} = \frac{\partial}{\partial w_m} v(\widetilde{w}) \frac{dw_m}{d\eta} = -v'(w_m) g_\tau(\widetilde{w}) \frac{dw_m}{d\eta} > 0$$

and analogously for $\partial v(\widetilde{w})/\partial\Delta\mu$. Inserting these expressions into Equation (A16) and recalling that $\rho > r$ we obtain

$$\frac{\partial\widetilde{w}}{\partial\lambda} = -\frac{r}{\rho - r} \frac{\partial v(\widetilde{w})}{\partial\lambda} > 0,$$

$$\frac{\partial\widetilde{w}}{\partial\eta} = -\frac{r}{\rho - r} \frac{\partial v(\widetilde{w})}{\partial\eta} < 0$$

and analogously for $\partial\widetilde{w}/\partial\Delta\mu$, which concludes the proof. ∎

Proof of Corollary 2: Using the definition of the limit on the credit line c, rewrite

$$rD = \overline{X}\left(\mu - \rho\frac{\widetilde{w}}{\eta}\right)$$

then

$$\frac{\partial rD}{\partial\Delta\mu} = -\overline{X}\frac{\rho}{\eta}\frac{r}{\rho - r}\frac{\partial v}{\partial\Delta\mu} < 0$$

$$\frac{\partial rD}{\partial\lambda} = -\overline{X}\frac{\rho}{\eta}\frac{r}{\rho - r}\frac{\partial v}{\partial\lambda} > 0$$

$$\frac{\partial D}{\partial\eta} = \underbrace{-\overline{X}\frac{\rho}{\eta}\frac{\partial\widetilde{w}}{\partial\eta}}_{<0} + \underbrace{\overline{X}\frac{\rho}{\eta^2}\widetilde{w}}_{>0}$$

making use of the results of Proposition 5. Similar computations can be carried out for the credit limit c; they yield ambiguous results. ∎

Proof of Proposition 7: We need to understand what value w_i of the continuation utility w is conducive of investment. Clearly there cannot be any investment at the boundary w_m: at that point the principal is forced to downsize to preserve incentive compatibility. So for any $c \geq 0$, $w_i > w_m$ – even when $c = 0$ because of the risk taking problem at w_m. To

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characterize w_i more precisely, note that the form of the HJB equation of the value function depends on whether there is investment; that is, on whether $w_t \geq w_i$. On the no-investment range (w_m, w_i) the function v_n satisfies

$$
\begin{aligned}
rv_n(w) = \mu dt + \sup_{di_t, dx_t, g_t} \Big\{ &- di_t + v_n'(w)(\rho w_t dt - di_t) \\
&+ (v_n(w) - w v_n'(w))dx_t + \frac{\beta^2}{2} v_n''(w)dt \Big\},
\end{aligned}
\tag{A18}
$$

while above w_i the function v_i follows

$$
\begin{aligned}
(r - g_t)v_i(w) = \mu dt - g_t c + \sup_{di_t, dx_t, g_t} \Big\{ &- di_t + v_1'(w)[(\rho - g_t)w_t dt - di_t] \\
&+ (v_i(w) - w v_i'(w))dx_t + \frac{\beta^2}{2} v_i''(w)dt \Big\}.
\end{aligned}
\tag{A19}
$$

We need to show there exists a solution to the differential Equation (A19). As before, any such solution may be written as the sum of the particular solution $v \equiv \frac{\mu - gc}{r - g}$ and one particular solution to the homogeneous equation

$$
rh(w) = \rho w h'(w) + \frac{\beta^2}{2} h''(w)
\tag{A20}
$$

as in the proof of Proposition 2. Let h_0 and h_1 denote the particular solutions to Equation A20 that satisfy $h_0(w_m) = 1, h_1(w_m) = 0, h_0'(w_m) = 0$ and $h_1'(w_m) = 1$. With these basis functions,

$$
v_i(w) = \frac{\mu - gc}{r - g} + b_0 h_0(w) + b_1 h_1(w), \quad w \in [w_m, \widetilde{w}]
$$

for some $\widetilde{w} \geq w_m$. The boundary conditions $v_i(w_m) = \pi := \frac{\Pi}{\overline{X}}$ implies

$$
\frac{\mu - gc}{r - g} + b_0 h_0(w_m) + b_1 h_1(w_m) = \pi \Rightarrow b_0 = \pi - \frac{\mu - gc}{r - g}
$$

and $v_i'(\widetilde{w}) = -1$ yields

$$
b_0 h_0'(\widetilde{w}) + b_1 h_1'(\widetilde{w}) = -1, \Rightarrow b_1 = -\frac{1}{h_1'(\widetilde{w})} \left[1 + \Big(\pi - \frac{\mu - gc}{r - g} \Big) h_0'(\widetilde{w}) \right].
$$

Therefore

$$
v_i(w) = v_i(w; \widetilde{w}) = \frac{\mu - gc}{r - g} + \Big(\pi - \frac{\mu - gc}{r - g} \Big) h_0(w) - \left[1 + \Big(\pi - \frac{\mu - gc}{r - g} \Big) h_0'(\widetilde{w}) \right] \frac{h_1(w)}{h_1'(\widetilde{w})}, \tag{A21}
$$

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which is also indexed by $\widetilde{w}$, and satisfies the boundary $v_i(w_m) = \pi$. The rest proceeds as in the proof of Proposition 2.

Second we combine these solution at the investment threshold w_i. That is,

$$v(w) = \begin{cases} v_n(w), & \text{for } w_m < w \le w_i; \text{ and} \\ v_{ni}(w), & \text{for } w_i \le w < \widetilde{w}. \end{cases}$$

with equality at w_i: the principal must be indifferent between investing and not, that is, $v_n(w_i) = v_i(w_i)$ and the function $v(w)$ must be $\mathcal{C}^2$. Re-arranging:

$$v(w_i) - w_i v'(w_i) = c \tag{A22}$$

More precisely, the function $v(w)$ must obey the termination condition at w_m, so $a_0 = \pi - \mu/r$ as in Proposition 2, and the boundary condition $v'(\widetilde{w}) = -1$, now given by $v_n(w)$. That is,

$$b_0 h_0'(\widetilde{w}) + b_1 h_1'(\widetilde{w}) = -1 \Rightarrow b_0 = -\frac{1 + b_1 h_1'(\widetilde{w})}{h_0'(\widetilde{w})}$$

We still have to determine b_1, a_1 and w_i, for which we may exploit continuity, smooth-pasting and super contact at w_i.

$$\frac{\mu}{r} + a_0 h_0(w_i) + a_1 h_1(w_i) = \frac{\mu - gc}{r - g} + b_0 h_0(w_i) + b_1 h_1(w_i)$$
$$a_0 h_0'(w_i) + a_1 h_1'(w_i) = b_0 h_0'(w_i) + b_1 h_1'(w_i)$$
$$a_0 h_0''(w_i) + a_1 h_1''(w_i) = b_0 h_0''(w_i) + b_1 h_1''(w_i)$$

$b_0 - a_0 = -\left[\frac{1 + b_1 h_1'(\widetilde{w})}{h_0'(\widetilde{w})} + \pi - \frac{\mu}{r}\right]$ and from the last two equations,

$$b_1 - a_1 = -(b_0 - a_0)\frac{h_0''(w_i)}{h_1''(w_i)} = \frac{h_0''(w_i)}{h_1''(w_i)}\left[\frac{1 + b_1 h_1'(\widetilde{w})}{h_0'(\widetilde{w})} + \pi - \frac{\mu}{r}\right]$$

Hence w_i is identified by

$$\frac{g(rc - \mu)}{r(r - g)} = -\left[\frac{1 + b_1 h_1'(\widetilde{w})}{h_0'(\widetilde{w})} + \pi - \frac{\mu}{r}\right]\left(h_0(w_i) - \frac{h_0''(w_i)}{h_1''(w_i)}h_1(w_i)\right)$$

To bound w_i from above we make use of the boundary condition at $\widetilde{w}$ and of the necessary condition for investment $v(\widetilde{w}) + \widetilde{w} > c$. Together they imply

$$w_i < \widetilde{w},$$

which is intuitive if we recall that an important benefit of investment is to delay payments to the agent. For this it must take place before that payment barrier is reached. ∎

Proof of Proposition 9:

We have from Lemma A1 that the boundary–value problem

$$rv(w) = \mu + \rho w v'(w) + \frac{\beta^2}{2} v''(w) \quad \text{for } w \in (w_m, \tilde{w})$$

$$v'(\tilde{w}) = -1 \quad \text{and} \quad v''(\tilde{w}) = 0 \tag{A23}$$

has a unique solution and that there is a unique $\tilde{w} \geq w_m$ such that $v(w_m) = \pi$, the required size–adjusted value at $w = w_m$. For each $\pi \geq 0$ we can then define $v(\cdot; \pi)$ as the solution to Problem (A23) such that $\tilde{w}(\pi)$ is chosen so as to satisfy the boundary condition $v(w_m; \pi) = \pi$ and $v(w; \pi) = v(\tilde{w}(\pi); \pi) - (w - \tilde{w}(\pi))$ for $w > \tilde{w}(\pi)$. Clearly the value of the firm is increasing in its liquidation value π; hence, we have that $v(w; 0) \leq v(w; \pi)$ for all $w \geq w_m$ and all $\pi \geq 0$ and from Lemma A2 we see that the mapping $\pi \mapsto \tilde{w}(\pi)$ is decreasing. This allows us to concentrate on the interval $[w_m, \tilde{w}(0)]$. Let $\bar{\pi}$ be the smallest boundary level such that $\tilde{w}(\pi) = w_m$, where the firm's liquidation value is so high that it is always immediately liquidated. Clearly $v(w; \pi) \leq \bar{\pi}$ for all $w \geq w_m$ and we also have, again from Lemma A2, that $-1 \leq v'(w; \pi) \leq v'(w_m; 0)$. In other words, the family $\mathcal{A}_{\bar{\pi}} := \{v(\cdot; \pi), \pi \in [0, \bar{\pi}]\}$ is a uniformly bounded collection of concave functions with uniformly bounded derivatives; thus, it is also uniformly equicontinuous.

Below we construct a sequence $\{v_n \in \mathcal{A}_{\bar{\pi}}, n \in \mathbb{N}\}$ and call on the Arzela–Ascoli theorem to show it must have a limit point. Said limit will be the solution to Problem (A23) subject to Condition 6.8. The aforementioned sequence is generated in the following way:

1. Set $\pi_1 = \bar{\pi}$, *exit flag*=0 and $v_1(\cdot) = v(\cdot; \pi_1)$.

2. Let $w_1^* = 0$ and compute $\kappa_1 := x_0 - 1 - \gamma + x_0 v_1(w_1^*) = x_0 - 1 - \gamma + x_0 \bar{\pi}$.

3. If $\kappa_1 < \pi$ restructuring is too costly, i.e. γ is too large relative to the other problem parameters, and *exit flag*=1

4. If $\kappa_1 \geq \pi$ we set $\pi_2 = \kappa_1$ and repeat the above process.

5. If *exit flag*=0 after the $n-1$ iteration, the n–th step of the iterative procedure consists in solving Problem (A23) subject to the condition $v_n(w_m) = \kappa_{n-1}$ as long as $\kappa_{n-1} \geq \pi$. Otherwise we deem reorganization too costly, set *exit flag*=1 and stop.

If the state *exit flag*=1 is never reached, we have the required sequence $\{v_n \in \mathcal{A}_{\overline{\pi}}, n \in \mathbb{N}\}$ and, by construction, its limit element $v_\infty \in \mathcal{A}_{\overline{\pi}}$ and it satisfies

$$v_\infty(w_m) = x_0 - 1 - \gamma + x_0 v_\infty(w_\infty^*),$$

as required.

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